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Stationary Distribution and Extinction of Stochastic Beddington-DeAngelis Predator-prey Model with Distributed Delay

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Abstract In this paper, we consider the dynamics of the stochastic Beddington-DeAngelis predator-prey model with distributed delay. First, we adopt the linear chain technique to transfer the stochastic system with strong kernel into an equivalent degenerated stochastic system made up four equations. Then we give the existence and uniqueness of the global positive solution. Next, sufficient conditions for persistence and extinction of two species are obtained. Particularly, the existence of the stationary distribution is established by constructing a suitable Lyapunov function. Finally, numerical simulations illustrate our theoretical results. It shows that the system still maintains the stability for the smaller white noises, but the stronger white noises will lead to the extinction of one or two species.

Keywords stochastic Beddington-DeAngelis predator-prey model, Distributed delay, Stationary distribution, Extinction.

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1. Introduction

Predator-prey systems describing the dynamic relationship between two species have long been and will continue to be the focus in an existing ecosystem. Predator-dependent functional response is the significant component describing the predator-prey relationship (see [1–3]). Particularly, the Beddington-DeAngelis functional response plays an important role in feeding over a range of predator-prey abundances [4–7]. Cantrell et al. [8] and Hwang [9] studied a classical predator-prey system with Beddington-DeAngelis response as follows:

$$\begin{aligned} \frac{dx}{dt} &= b_1 x\left(1-\frac{x}{k}\right) - \frac{a_{12}xy}{m_1+m_2x+m_3y}, \\ \frac{dy}{dt} &= -b_2 y + \frac{a_{21}xy}{m_1+m_2x+m_3y}, \end{aligned} \tag{1.1}$$

where $x(t)$ and $y(t)$ denote the prey and predator densities at time t respectively. And b_1 and k are intrinsic growth rate of the prey and carrying capacity of the envi-

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ronment in the absence of predator, b_2 is the death rate of the predator. The predator consumes the prey with Beddington-DeAngelis functional response $\frac{a_{12}xy}{m_1+m_2x+m_3y}$ and contributes to its growth rate $\frac{a_{21}xy}{m_1+m_2x+m_3y}$. All parameters are assumed to be positive.

It is known that [8,9] , if $b_2 \geq \frac{a_{21}k}{m_1+km_2}$, then system (1.1) has two boundary equilibria $(0,0),(k,0)$. And $(k,0)$ is globally asymptotically stable. If $0 < b_2 < \frac{a_{21}k}{m_1+km_2}$, then system (1.1) has three nonnegative equilibria $(0,0)$, $(k,0)$ and (x_*,y_*), where (x_*,y_*) is positive and satisfies the following equation:

$$\frac{b_1a_{21}}{m_3ka_{12}}x_*^2 - \left(b_2m_2 + \frac{b_1a_{21}}{m_3a_{12}} - a_{21}\right)x_* - b_2m_1 = 0,$$
$$y^* = \frac{(a_{21}-b_2m_2)x^* - b_2m_1}{b_2m_3}.$$

And the local and global asymptotic stabilities of (x_*,y_*) coincide.

On the other hand, the growth of biological organisms may depend on the population density of previous time. Distributed delay has been widely introduced into equations used in mathematical biology (see [10–13]). Some authors (see e.g. [10,14]) studied the stability and the bifurcation of a predator-prey model with distributed delay. A classical Beddington-DeAngelis predator-prey system with distributed delay is as follows:

$$\begin{cases} \dfrac{dx}{dt} = b_1x\left(1-\frac{x}{k}\right) - \frac{a_{12}xy}{m_1+m_2x+m_3y}, \\ \dfrac{dy}{dt} = -b_2y + a_{21}\int_{-\infty}^{t} \frac{K(t-s)x(s)y(s)}{m_1+m_2x(s)+m_3y(s)}ds. \end{cases} \tag{1.2}$$

In addition, in the natural environment, the growth rate of biological population is inevitably affected by white noises. In some cases, white noise will have a huge impact on the size of the biological population and the evolution of biological population. Some researchers [15–18] studied stationary distribution and extinction of a kind of predator-prey model with stochastic disturbance.

In this article, we assume that the intrinsic growth rates b_1 and b_2 of the prey and the predator are disturbed with:

$$b_1 \to b_1 + \alpha_1\dot{B}_1(t),\ b_2 \to b_2 + \alpha_2\dot{B}_2(t),$$

where $B_1(t),\ B_2(t)$ are independent Brownian motion, α_1^2 and α_2^2 represent the intensity of the white noises, respectively. Then system (1.2) can be transformed into the following stochastic model:

$$\begin{cases} dx = \left(b_1x\left(1-\frac{x}{k}\right) - \frac{a_{12}xy}{m_1+m_2x+m_3y}\right)dt + \alpha_1xdB_1(t), \\ dy = \left(-b_2y + a_{21}\int_{-\infty}^{t} \frac{K(t-s)x(s)y(s)}{m_1+m_2x(s)+m_3y(s)}ds\right)dt - \alpha_2ydB_2(t). \end{cases} \tag{1.3}$$

We focus on two well-known ones: the strong kernel and the weak kernel, respectively, represented by

$$(1)\ K(t) = \sigma^2te^{-\sigma t}\ ,\ (2)\ K(t) = \sigma e^{-\sigma t}.$$

These two kinds of kernels have been widely used in many fields of biological system, such as population system [19], neutral network [20,21] and epidemiology [22].

In this article, we mainly focus on establishing sharp sufficient conditions for the existence of a stationary distribution of system (1.3). As far as we know, there have been some results on the stationary distribution of stochastic system with discrete delay (see [23, 24]). However, there is only a little work on the existence of the stationary distribution of stochastic system with distributed delay, for instance, a Logistic model with distributed delay [25], a cooperative Lotka-Volterra system with distributed delay [26], stochastic recurrent neural networks with discrete and distributed delay [27], a predator-prey system with distributed delay [28].

The rest of this paper is organized as follows. In the next section, we give the main results of the stochastic system (1.3) in the strong kernel case. Sections 3-5 give the detailed proof. In Section 6, we give the results in the weak kernel case. Finally, numerical simulations are given to verify our results.

Throughout this paper, unless otherwise specified, we suppose($\Omega, \{\mathscr{F}_t\}_{t\geq 0}, P$) is a complete probability space with $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Define

$$\mathbb{R}^4_+ = \{\ (x,y,u,w) \in \mathbb{R}^4 : x>0, y>0, u>0, w>0\ \}.$$

2. Main results in the strong kernel case

Define

$$\begin{aligned} u(t) &= \int_{-\infty}^{t} \sigma^2 (t-s) e^{-\sigma(t-s)} \frac{x(s)y(s)}{m_1+m_2x(s)+m_3y(s)} ds, \\ w(t) &= \int_{-\infty}^{t} \sigma e^{-\sigma(t-s)} \frac{x(s)y(s)}{m_1+m_2x(s)+m_3y(s)} ds. \end{aligned}$$

Then by the linear chain technique, system (1.3) can be transformed into the following system:

$$\begin{cases} dx &= \left(b_1 x\left(1-\frac{x}{k}\right) - \frac{a_{12}xy}{m_1+m_2x+m_3y}\right)dt + \alpha_1 x dB_1(t), \\ dy &= (-b_2 y + a_{21} u)dt - \alpha_2 y dB_2(t), \\ du &= \sigma(w-u)dt, \\ dw &= \sigma\left(\frac{xy}{m_1+m_2x+m_3y} - w\right)dt. \end{cases} \tag{2.1}$$

Theorem 2.1. *For any given initial value $(x(0), y(0), u(0), w(0)) \in \mathbb{R}^4_+$, there exists a unique positive solution $(x(t), y(t), u(t), w(t))$ of system (2.1) on $t \geq 0$ and the solution will remain in $\mathbb{R}^4_+$ with probability one.*

Theorem 2.2. *Let $(x(t), y(t), u(t), w(t))$ be a solution of system (2.1) with any initial value $(x(0), y(0), u(0), w(0)) \in \mathbb{R}^4_+$. Then the following results are true:*

(1) If $b_1 < \frac{\alpha_1^2}{2}$, then $\lim_{t\to\infty} x(t) = 0$ a.s.

(2) If $b_1 > \frac{\alpha_1^2}{2}$, then for almost all $\omega \in \Omega$, we have

$$\limsup_{t\to\infty} \frac{1}{t} \ln\left(\sqrt{\frac{k}{a_{21}b_2(m_1+m_2k)}}y(t) + \frac{1}{\sigma}u(t) + \frac{1}{\sigma}w(t)\right) \leq \mu,$$

where $\mu = \min\{b_2, \sigma\}(\sqrt{R_0} - 1)I_{\{R_0 \leq 1\}} + \max\{b_2, \sigma\}(\sqrt{R_0} - 1)I_{\{R_0 \geq 1\}} + \alpha_1 b_2 \sqrt{\frac{R_0}{2b_1}}$ *and* $R_0 = \frac{a_{21}k}{b_2(m_1+m_2k)}$. *Especially, if* $\mu < 0$, *then the predator population* y *will die out exponentially with probability one, i,e. ,*

$$\lim_{t\to\infty} y(t) = 0 \quad a.s.$$

Moreover, the distribution of $x(t)$ *converges weakly to the measure which has the density*

$$\pi(u) = Q\alpha_1^{-2} u^{-2+\frac{2b_1}{\alpha_1^2}} e^{-\frac{2b_1}{k\alpha_1^2}}, u \in (0, \infty),$$

where $Q = [\alpha_1^{-2}(\frac{k\alpha_1^2}{2b_1})^{-1+\frac{2b_1}{\alpha_1^2}} \Gamma(\frac{2b_1}{\alpha_1^2} - 1)]^{-1}$ *satisfying* $\int_0^\infty \pi(u)du = 1$.

Theorem 2.3. *Assume that* $b_1 > \frac{\alpha_1^2}{2}, b_2 > \frac{\alpha_2^2}{2}$, *and* $R_0^S = \frac{a_{21}k(1-\frac{\alpha_1^2}{2b_1})^4}{(m_1+m_2k)(b_2+\frac{\alpha_2^2}{2})} > 1$. *Then system (2.1) exists a solution* $P^*(t) \triangleq (x^*(t), y^*(t), u^*(t), w^*(t))$, *which is a stationary Markov process.*

3. Materials and method

3.1. Proof of Theorem 2.1

Since the coefficients of the system (2.1) satisfy local Lipschitz condition for any given initial value $(x(0), y(0), u(0), w(0)) \in \mathbb{R}_+^4$, there exists a unique maximal local solution $(x(t), y(t), u(t), w(t))$ on $t \in (0, \tau_e)$, where τ_e is the explosion time. Next, we claim that $\tau_e = \infty$. We employ similar method of Theorem 3.1 of Mao *et al*. [29]. The key step is to construct a nonnegative C^2-function $\tilde{\mathcal{V}} : \mathbb{R}_+^4 \to \mathbb{R}_+$ such that

$$\liminf_{k\to\infty,(x,y,u,w)\in\mathbb{R}_+^4\setminus D_k} \tilde{\mathcal{V}}(x, y, u, w) = +\infty \text{ and } L\tilde{\mathcal{V}}(x, y, u, w) \leq M,$$

where $D_k = (\frac{1}{k}, k) \times (\frac{1}{k}, k) \times (\frac{1}{k}, k) \times (\frac{1}{k}, k)$, and M is a positive constant.

Define

$$\begin{aligned}\tilde{\mathcal{V}}(x, y, u, w) = &\frac{1}{a_{12}}(x - 1 - \ln x) + \frac{1}{a_{21}}(y - 1 - \ln y) + \frac{1}{\sigma}(u - 1 - \ln u)\\ &+ \frac{1}{\sigma}(w - 1 - \ln w).\end{aligned}$$

The nonnegativity of $\tilde{\mathcal{V}}(x, y, u, w)$ can be verified from

$$x - 1 - \ln x \geq 0, \text{ for } x > 0.$$

First

$$\inf_{(x,y,u,w)\in\mathbb{R}_+^4\setminus D_k} \tilde{\mathcal{V}}(x, y, u, w) \to +\infty, \text{ as } k \to +\infty,$$

which is clearly established since

$$x - 1 - \ln x \to +\infty, \ as\ x \to 0^+, \text{ and } x - 1 - \ln x \to +\infty \text{ as } x \to +\infty.$$

Next we claim that there exists a positive constant M such that $L\tilde{\mathcal{V}}(x,y,u,w) \le M$. Making use of Itô formula, we obtain

$$d\tilde{\mathcal{V}}(x,y,u,w) = L\tilde{\mathcal{V}}(x,y,u,w)dt + \alpha_1(x-1)dB_1(t) - \alpha_2(y-1)dB_2(t),$$

where $L : \mathbb{R}_+^4 \to \mathbb{R}_+$ is defined by

$$\begin{aligned}
L\tilde{\mathcal{V}}(x,y,u,w) &= \tfrac{1}{a_{12}}(x-1)\left(b_1\left(1-\tfrac{x}{k}\right) - \tfrac{a_{12}y}{m_1+m_2x+m_3y}\right) + \tfrac{1}{a_{21}}\left(1-\tfrac{1}{y}\right)(-b_2y+a_{21}u)\\
&\quad + \left(1-\tfrac{1}{w}\right)\left(\tfrac{xy}{m_1+m_2x+m_3y} - w\right) + \left(1-\tfrac{1}{u}\right)(w-u) + \tfrac{\alpha_1^2}{2a_{12}} + \tfrac{\alpha_2^2}{2a_{21}}\\
&= \tfrac{\alpha_1^2}{2a_{12}} + \tfrac{\alpha_2^2}{2a_{21}} + \tfrac{b_2}{a_{21}} + 2 + \tfrac{b_1}{a_{12}}\left(-\tfrac{x^2}{k} + \left(\tfrac{1}{k}+1\right)x - 1\right) + \tfrac{y}{m_1+m_2x+m_3y}\\
&\le \tfrac{\alpha_1^2}{2a_{12}} + \tfrac{\alpha_2^2}{2a_{21}} + \tfrac{b_2}{a_{21}} + 2 + \tfrac{1}{m_3} + \tfrac{b_1}{a_{12}}\left(-\tfrac{x^2}{k} + \left(\tfrac{1}{k}+1\right)x - 1\right)\\
&\le M,
\end{aligned}$$

where

$$M = \frac{\alpha_1^2}{2a_{12}} + \frac{\alpha_2^2}{2a_{21}} + \frac{b_2}{a_{21}} + 2 + \frac{1}{m_3} + \max_{x\in(0,+\infty)}\{\frac{b_1}{a_{12}}\left(-\frac{x^2}{k} + \left(\frac{1}{k}+1\right)x - 1\right)\}.$$

3.2. Proof of Theorem 2.2

To prove Theorem 2.2, we first give the definition of the persistence and the extinction and a lemma from [30]:

Definition 3.1. System (2.1) is said to be extinct if $\limsup_{t\to+\infty} y(t) = 0$. System (2.1) is said to be persistent in mean if $\liminf_{t\to+\infty} \frac{1}{t}\int_0^t y(s)ds > 0$.

Consider the following 1-dimensional stochastic differential equation

$$dX(t) = b_1X\left(1-\frac{X}{k}\right)dt + \alpha_1XdB_1(t). \tag{3.1}$$

Lemma 3.1. *([31]) If $b_1 < \frac{\alpha_1^2}{2}$, then $\lim_{t\to\infty} X(t) = 0$ a.s.; if $b_1 > \frac{\alpha_1^2}{2}$, then Eq.(3.1) has a stationary solution $\widetilde{X}(t)$, which has the density*

$$\pi(u) = Q\alpha_1^{-2}u^{-2+\frac{2b_1}{\alpha_1^2}}e^{-\frac{2b_1}{k\alpha_1^2}},\ u\in(0,\infty),$$

where $Q = [\alpha_1^{-2}(\frac{k\alpha_1^2}{2b_1})^{-1+\frac{2b_1}{\alpha_1^2}}\Gamma(\frac{2b_1}{\alpha_1^2}-1)]^{-1}$ such that $\pi(u)$ satisfies $\int_0^\infty \pi(u)du = 1$.

Proof of Theorem 2.2 (1) If $b_1 < \frac{\alpha_1^2}{2}$, obviously,

$$\begin{aligned}
d\ln x &= \left(\left(b_1 - \tfrac{b_1x}{k}\right) - \tfrac{a_{12}y}{m_1+m_2x+m_3y} - \tfrac{\alpha_1^2}{2}\right)dt + \alpha_1dB_1(t)\\
&\le \left(b_1 - \tfrac{\alpha_1^2}{2}\right)dt + \alpha_1dB_1(t).
\end{aligned}$$

For $b_1 < \frac{\alpha_1^2}{2}$, we have $\lim_{t\to\infty} \ln x(t) = -\infty$, and so $\lim_{t\to\infty} x(t) = 0$ a.s. Thus, the proof of Theorem 2.2(1) is completed.

(2) Since for any initial value $(x(0), y(0), u(0), w(0)) \in \mathbb{R}_+^4$, the solution of system (2.1) is positive, we get

$$dx(t) \le b_1 x\left(1 - \frac{x}{k}\right)dt + \alpha_1 x dB_1(t).$$

Let $X(t)$ be the solution of SDE(3.1) with the initial value $X(0) = x(0) > 0$. Then applying the comparison theorem of 1-dimensional stochastic differential equation [32], we have $x(t) \le X(t)$ for any $t \ge 0$ a.s.

Moreover by the proof of Theorem 2.2 of Liu [28], we have that,

$$\begin{aligned}
\textstyle\int_0^\infty u\pi(u)du &= \frac{k\left(b_1 - \frac{\alpha_1^2}{2}\right)}{b_1},\\
\textstyle\int_0^\infty u^2\pi(u)du &= \frac{k^2\left(b_1 - \frac{\alpha_1^2}{2}\right)}{b_1},\\
\textstyle\int_0^\infty (u-k)^2\pi(u)du &= \frac{k^2\alpha_1^2}{2b_1}.
\end{aligned}$$

We define a $C^2 - function\ \mathcal{V} : \mathbb{R}_+^3 \to \mathbb{R}_+$ by

$$\mathcal{V}(y, u, w) = \delta_1 y(t) + \delta_2 u(t) + \delta_2 w(t),$$

where δ_1, δ_2 are positive constants to be determined later. Making use of Itô formula to differentiate $\ln \mathcal{V}$ yields

$$d(\ln \mathcal{V}) = L(\ln \mathcal{V})dt - \frac{\delta_1 \alpha_2 y}{\mathcal{V}} dB_2(t),$$

where

$$\begin{aligned}
L(\ln \mathcal{V}) &= \tfrac{\delta_1}{\mathcal{V}}(a_{21}u - b_2 y) + \tfrac{\delta_2}{\mathcal{V}}\left(\sigma w - \sigma u + \tfrac{\sigma xy}{m_1 + m_2 x + m_3 y} - \sigma w\right) - \tfrac{\delta_1^2\alpha_2^2 y^2}{2\mathcal{V}^2}\\
&\le \tfrac{\delta_1}{\mathcal{V}}(a_{21}u - b_2 y) + \tfrac{\delta_2}{\mathcal{V}}\left(\tfrac{\sigma xy}{m_1 + m_2 x + m_3 y} - \sigma u\right)\\
&\le \tfrac{\delta_1}{\mathcal{V}}(a_{21}u - b_2 y) + \tfrac{\delta_2}{\mathcal{V}}\left(\tfrac{\sigma xy}{m_1 + m_2 x} - \sigma u\right)\\
&= \tfrac{\delta_1}{\mathcal{V}}(a_{21}u - b_2 y) + \tfrac{\delta_2}{\mathcal{V}}\left(\tfrac{\sigma xy}{m_1 + m_2 x} - \sigma u + \tfrac{\sigma ky}{m_1 + m_2 k} - \tfrac{\sigma ky}{m_1 + m_2 k}\right)\\
&= \tfrac{\sigma\delta_2 y}{\mathcal{V}}\left(\tfrac{x}{m_1 + m_2 x} - \tfrac{k}{m_1 + m_2 k}\right) + \tfrac{1}{\mathcal{V}}\{\tfrac{\sigma\delta_2 ky}{m_1 + m_2 k} + a_{21}\delta_1 u - (\delta_1 b_2 y + \delta_2\sigma u)\}.
\end{aligned}$$

We define

$$M_0 = \begin{bmatrix} 0 & \frac{a_{21}}{b_2} & 0\\ \frac{k}{m_1 + m_2 k} & 0 & 0\\ 0 & 0 & 1 \end{bmatrix}.$$

Making use of spectral decomposition, we let $(\delta_1 b_2, \delta_2\sigma, \delta_2)M_0 = \sqrt{R_0}(\delta_1 b_2, \delta_2\sigma, \delta_2)$, where $R_0 = \frac{a_{21}k}{b_2(m_1+m_2k)}$.

$$
\begin{aligned}
L(\ln\mathcal{V}) &\leq \tfrac{\sigma\delta_2 y}{\mathcal{V}}\left(\tfrac{x}{m_1+m_2x} - \tfrac{k}{m_1+m_2k}\right) + \tfrac{1}{\mathcal{V}}(\delta_1 b_2, \delta_2\sigma, \delta_2)(M_0(y,u,w)^T - (y,u,w)^T)\\
&= \tfrac{\sigma\delta_2 y}{\mathcal{V}}\left(\tfrac{x}{m_1+m_2x} - \tfrac{k}{m_1+m_2k}\right) + \tfrac{1}{\mathcal{V}}(\sqrt{R_0}-1)(\delta_1 b_2 y + \sigma\delta_2 u + \delta_2 w)\\
&= \tfrac{\sigma\delta_2 m_1 y}{\mathcal{V}(m_1+m_2x)(m_1+m_2k)}(x-k) + \tfrac{1}{\mathcal{V}}(\sqrt{R_0}-1)(\delta_1 b_2 y + \sigma\delta_2 u + \delta_2 w)\\
&\leq \tfrac{y\sigma\delta_2}{\mathcal{V}(m_1+m_2k)}(X-k) + \tfrac{1}{\mathcal{V}}(\sqrt{R_0}-1)(\delta_1 b_2 y + \sigma\delta_2 u + \delta_2 w)\\
&\leq \tfrac{\delta_2\sigma}{\delta_1(m_1+m_2k)}|X-k| + \tfrac{1}{\mathcal{V}}(\sqrt{R_0}-1)(\delta_1 b_2 y + \sigma\delta_2 u + \delta_2 w)\\
&\leq \min\{b_2,\sigma\}(\sqrt{R_0}-1)I_{\{R_0\leq 1\}} + \max\{b_2,\sigma\}(\sqrt{R_0}-1)I_{\{R_0\geq 1\}}\\
&\quad + \tfrac{\delta_2\sigma}{\delta_1(m_1+m_2k)}|X-k|.
\end{aligned}
$$

Setting $\gamma = \min\{b_2,\sigma\}(\sqrt{R_0}-1)I_{\{R_0\leq 1\}} + \max\{b_2,\sigma\}(\sqrt{R_0}-1)I_{\{R_0\geq 1\}}$, we have

$$
d(\ln\mathcal{V}) \leq \{\gamma + \tfrac{\delta_2\sigma}{\delta_1(m_1+m_2k)}|X-k|\}dt - \tfrac{\delta_1\alpha_2 y}{\mathcal{V}}dB_2(t).
$$

Integrating from 0 to t and dividing by t for the formula above, we get that

$$
\tfrac{\ln\mathcal{V}(t)}{t} \leq \tfrac{\ln\mathcal{V}(0)}{t} + \gamma + \tfrac{\delta_2\sigma}{t\delta_1(m_1+m_2k)}\int_0^t |X(s)-k|ds - \tfrac{1}{t}\int_0^t \tfrac{\delta_1\alpha_2 y(s)}{\mathcal{V}(s)}dB_2(s), \tag{3.2}
$$

where $M(t) \triangleq \int_0^t \frac{\delta_1\alpha_2 y(s)}{\mathcal{V}(s)}dB_2(s)$ is a local martingale. The strong law number [29] for local martingale leads to

$$
\lim_{t\to\infty}\frac{M(t)}{t} = 0 \ \ a.s. \tag{3.3}
$$

Since X(t) is ergodic and $\int_0^\infty u\pi(u)du = \frac{k(b_1-\frac{\alpha_1^2}{2})}{b_1} < \infty$, we have that by Hölder inequality,

$$
\lim_{t\to\infty}\frac{1}{t}\int_0^t |X(s)-k|ds = \int_0^\infty |u-k|\pi(u)du \leq (\int_0^\infty (u-k)^2\pi(u)du)^{\frac{1}{2}}. \tag{3.4}
$$

Taking the superior limit on the both sides of (3.2) and substituting $\delta_1 = \frac{k}{b_2(m_1+m_2k)}$, $\delta_2 = \frac{\sqrt{R_0}}{\sigma}$ into the following inequalities, together with (3.3) and (3.4), we obtain that

$$
\begin{aligned}
\limsup_{t\to\infty}\frac{\ln\mathcal{V}(t)}{t} &\leq \lim_{t\to\infty}\frac{\ln\mathcal{V}(0)}{t} + \gamma + \frac{\delta_2\sigma}{\delta_1(m_1+m_2k)}\lim_{t\to\infty}\frac{1}{t}\int_0^t |X(s)-k|ds\\
&\leq \gamma + \frac{\delta_2\sigma}{\delta_1(m_1+m_2k)}\sqrt{\frac{k^2\alpha_1^2}{2b_1}}\\
&= \gamma + \alpha_1 b_2\left(\frac{R_0}{2b_1}\right)^{\frac{1}{2}} := \mu \ a.s.,
\end{aligned}
$$

which is required assertion. In addition, when $\mu < 0$, it's obvious to get

$$\limsup_{t\to\infty} \frac{\ln y(t)}{t} < 0,$$

which also means $\lim_{t\to\infty} y(t) = 0\, a.s.$ That is to say, the predator population will reduce to zero exponentially with probability one.

3.3. Proof of Theorem 2.3

Consider the integral equation:

$$X(t) = X(t_0) + \int_{t_0}^{t} b(s, X(s))ds + \sum_{r=1}^{k} \int_{t_0}^{t} \sigma_r(s, X(s))dB_r(s), \quad t \geq t_0 \geq 0. \tag{3.1}$$

Lemma 3.2. *(see [33]). Suppose that the coefficients of (3.1) are independent of t and satisfy the following conditions for some constant B:*

$$\begin{aligned} &|b(s,x) - b(s,y)| + \textstyle\sum_{r=1}^{k} |\sigma_r(s,x) - \sigma_r(s,y)| \leq B|x-y|, \\ &|b(s,x)| + \textstyle\sum_{r=1}^{k} |\sigma_r(s,x)| \leq B(1+|x|), \end{aligned} \tag{3.2}$$

in U_R for every $R > 0$, and that there exists a non-negative $C^2 - function$ $V(x)$ in $\mathbb{R}^l$ such that

$$LV(x) \leq -1 \textit{ outside some compact set}.$$

Then system (3.1) exists a solution, which is a stationary Markov process.

Remark 3.1. The condition (3.2) can be replaced by the global existence of the solutions of (3.1) according to remark 5 of Xu [34].

Lemma 3.3. *For any $\xi > 0$, $\xi + \frac{3}{4}\xi(1-\xi) \leq \xi^{\frac{1}{4}}$.*

Proof. Setting $t = \xi^{\frac{1}{4}}$, we get

$$\begin{aligned} &\xi + \tfrac{3}{4}\xi(1-\xi) - \xi^{\frac{1}{4}} \\ &= \tfrac{1}{4}(7t^4 - 3t^8 - 4t) \\ &= -\tfrac{t}{4}(3t^7 - 7t^3 + 4) \\ &= -\tfrac{t}{4}(t-1)^2(3t^5 + 6t^4 + 9t^3 + 12t^2 + 8t + 4). \end{aligned}$$

Obviously, we have $\xi + \frac{3}{4}\xi(1-\xi) - \xi^{\frac{1}{4}} \leq 0$ for any $t > 0$.

Proof of Theorem 2.3. We will prove that system (2.1) exists a stationary distribution. By Theorem 2.1, we have obtained that system (2.1) has a global positive solution. By Lemma 3.2, it is sufficient to construct a non-negative C^2-function $V^*(x,y,u,w)$ and a closed set $\Omega \subset \mathbb{R}_+^4$ such that

$$LV^*(x,y,u,w) \leq -1, \text{ on } (x,y,u,w) \in \mathbb{R}_+^4/\Omega.$$

Since the construction of $V^*(x,y,u,w)$ is very complex, we construct it in four steps.

Define

$$\begin{aligned}\widetilde{V}_1(x,y,u,w) &= \left(-\ln y + \tfrac{m_2}{b_1}x - c_1\ln u - c_2\ln w\right) + 4\sqrt[4]{a_{21}c_1c_2\sigma^2 k}\ \left(\tfrac{3x}{4kb_1} - \tfrac{\ln x}{b_1}\right)\\ &\triangleq V_1 + V_2,\end{aligned}$$

where c_1, c_2 are positive, and will be chosen later.

Note that $m_2 x\left(1-\frac{x}{k}\right) \le m_2 k - m_2 x$.

By Itô formula, we have

$$\begin{aligned}LV_1 &= b_2 + \tfrac{\alpha_2^2}{2} - \tfrac{a_{21}u}{y} + m_2x\left(1-\tfrac{x}{k}\right) - \tfrac{a_{12}m_2xy}{b_1(m_1+m_2x+m_3y)}\\ &\quad -\sigma\left(\tfrac{c_1w}{u} + \tfrac{c_2xy}{w(m_1+m_2x+m_3y)} - c_1 - c_2\right)\\ &\le \left(b_2 + \tfrac{\alpha_2^2}{2} + (c_1+c_2)\sigma + m_2x\left(1-\tfrac{x}{k}\right)\right) - \left(\tfrac{a_{21}u}{y} + \tfrac{c_1w\sigma}{u} + \tfrac{c_2\sigma xy}{w(m_1+m_2x+m_3y)}\right)\\ &\le \left(b_2 + \tfrac{\alpha_2^2}{2} + (c_1+c_2)\sigma + m_2k\right) - \left(m_2x + \tfrac{a_{21}u}{y} + \tfrac{c_1w\sigma}{u} + \tfrac{c_2\sigma xy}{w(m_1+m_2x+m_3y)}\right)\\ &\le \left(b_2 + \tfrac{\alpha_2^2}{2} + (c_1+c_2)\sigma + (m_1+m_2k)\right) + m_3y\\ &\quad - \left(\tfrac{a_{21}u}{y} + \tfrac{c_1w\sigma}{u} + \tfrac{c_2\sigma xy}{w(m_1+m_2x+m_3y)} + (m_1+m_2x+m_3y)\right)\\ &\le \left(b_2 + \tfrac{\alpha_2^2}{2} + (c_1+c_2)\sigma + (m_1+m_2k)\right) - 4\sqrt[4]{xa_{21}c_1c_2\sigma^2} + m_3y.\end{aligned}$$

Applying Itô formula and Lemma 3.3, we obtain that

$$\begin{aligned}LV_2 &= 4\sqrt[4]{a_{21}c_1c_2\sigma^2k}\ \left(\left(\tfrac{3x}{4k}-1\right)\left(1-\tfrac{x}{k}\right) - \tfrac{3a_{12}xy}{4kb_1(m_1+m_2x+m_3y)} + \tfrac{a_{12}y}{b_1(m_1+m_2x+m_3y)} + \tfrac{\alpha_1^2}{2b_1}\right)\\ &\le 4\sqrt[4]{a_{21}c_1c_2\sigma^2k}\ \left(\tfrac{x}{k} + \tfrac{3x}{4k}\left(1-\tfrac{x}{k}\right) - \left(1-\tfrac{\alpha_1^2}{2b_1}\right) + \tfrac{a_{12}y}{b_1(m_1+m_2x+m_3y)}\right)\\ &\le 4\sqrt[4]{a_{21}c_1c_2\sigma^2k}\ \left(\sqrt[4]{\tfrac{x}{k}} + \tfrac{a_{12}y}{b_1m_1} - \left(1-\tfrac{\alpha_1^2}{2b_1}\right)\right)\\ &= 4\sqrt[4]{xa_{21}c_1c_2\sigma^2} + 4\sqrt[4]{a_{21}c_1c_2\sigma^2k}\,\tfrac{a_{12}y}{b_1m_1} - 4\sqrt[4]{a_{21}c_1c_2\sigma^2k}\ \left(1-\tfrac{\alpha_1^2}{2b_1}\right).\end{aligned}$$

Taking $c_1 = c_2 = \frac{a_{21}k(1-\frac{\alpha_2^2}{2b_1})^4}{(m_1+m_2k)\sigma}$, we have that

$$\begin{aligned}L\widetilde{V}_1 &\le \left(b_2 + \tfrac{\alpha_2^2}{2} + (c_1+c_2)\sigma + (m_1+m_2k)\right) - 4\sqrt[4]{a_{21}c_1c_2\sigma^2k}(1-\tfrac{\alpha_1^2}{2b_1})\\ &\quad + (m_3 + 4\sqrt[4]{a_{21}c_1c_2\sigma^2k}\tfrac{a_{12}}{b_1m_1})y\\ &= -(\sqrt[4]{R_0^s}-1)\left(b_2+\tfrac{\alpha_2^2}{2}\right) + \left(m_3 + 4\sqrt[4]{a_{21}c_1c_2\sigma^2k}\tfrac{a_{12}}{b_1m_1}\right)y\\ &:= -\lambda_0 + \left(m_3 + 4\sqrt[4]{a_{21}c_1c_2\sigma^2k}\tfrac{a_{12}}{b_1m_1}\right)y,\end{aligned} \tag{3.3}$$

where $R_0^S = \frac{a_{21}k(1-\frac{\alpha_1^2}{2b_1})^4}{(m_1+m_2k)(b_2+\frac{\alpha_2^2}{2})}$, $\lambda_0 = (\sqrt[4]{R_0^s}-1)\left(b_2+\frac{\alpha_2^2}{2}\right) > 0$, since $R_0^s > 1$.

Selecting

$$\widetilde{V}_2(x,y,u,w) = \widetilde{V}_1 + \tfrac{1}{b_2}\left(m_3 + 4\sqrt[4]{a_{21}c_1c_2\sigma^2k}\tfrac{a_{12}}{b_1m_1}\right)\left(y + \tfrac{a_{21}u}{\sigma} + \tfrac{a_{21}w}{\sigma}\right),$$

while

$$L\left(y+\tfrac{a_{21}w}{\sigma}+\tfrac{a_{21}u}{\sigma}\right)=-b_2y+\tfrac{a_{21}xy}{m_1+m_2x+m_3y},$$

we have

$$L\widetilde{V}_2\le -\lambda_0+\frac{a_{21}}{b_2}\left(m_3+4\sqrt[4]{a_{21}c_1c_2\sigma^2k}\frac{a_{12}}{b_1m_1}\right)\frac{xy}{m_1+m_2x+m_3y}. \tag{3.4}$$

Choose

$$\widetilde{V}_3(x,y,u,w)=\frac{1}{1+\theta}\left(x+\frac{a_{12}y}{4a_{21}}+\frac{a_{12}w}{\sigma}+\frac{a_{12}u}{2\sigma}\right)^{1+\theta},$$

where $\theta>0$ and will be defined later.

$$\begin{aligned}
L\widetilde{V}_3 &= \left(x+\tfrac{a_{12}y}{4a_{21}}+\tfrac{a_{12}w}{\sigma}+\tfrac{a_{12}u}{2\sigma}\right)^{\theta}\left(b_1x\left(1-\tfrac{x}{k}\right)-\tfrac{a_{12}b_2y}{4a_{21}}-\tfrac{a_{12}w}{2}-\tfrac{a_{12}u}{4}\right)\\
&\quad+\tfrac{\theta}{2}\left(x+\tfrac{a_{12}y}{4a_{21}}+\tfrac{a_{12}w}{\sigma}+\tfrac{a_{12}u}{2\sigma}\right)^{\theta-1}\left(\alpha_1^2x^2+(\tfrac{a_{12}\alpha_2}{4a_{21}})^2y^2\right)\\
&\le b_1x\left(x+\tfrac{a_{12}y}{4a_{21}}+\tfrac{a_{12}u}{2\sigma}+\tfrac{a_{12}w}{\sigma}\right)^{\theta}-\tfrac{b_1}{k}x^{\theta+2}-\tfrac{a_{12}^{\theta+1}b_2}{4^{\theta+1}a_{21}^{\theta+1}}y^{\theta+1}-\tfrac{a_{12}^{\theta+1}}{2^{\theta+2}\sigma^{\theta}}u^{\theta+1}\\
&\quad-\tfrac{a_{12}^{\theta+1}}{2\sigma^{\theta}}w^{\theta+1}+\tfrac{\theta}{2}\left(x+\tfrac{a_{12}y}{4a_{21}}+\tfrac{a_{12}u}{2\sigma}+\tfrac{a_{12}w}{\sigma}\right)^{\theta-1}\left(\alpha_1^2x^2+\left(\tfrac{a_{12}\alpha_2}{4a_{21}}\right)^2y^2\right)\\
&\le -\tfrac{b_1}{2k}x^{\theta+2}-\tfrac{a_{12}^{\theta+1}b_2}{2\times4^{\theta+1}a_{21}^{\theta+1}}y^{\theta+1}-\tfrac{a_{12}^{\theta+1}}{4\sigma^{\theta}}w^{\theta+1}-\tfrac{a_{12}^{\theta+1}}{2^{\theta+3}\sigma^{\theta}}u^{\theta+1}+B,
\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}
B &= \tfrac{\theta}{2}\left(x+\tfrac{a_{12}y}{4a_{21}}+\tfrac{a_{12}u}{2\sigma}+\tfrac{a_{12}w}{\sigma}\right)^{\theta-1}\left(\alpha_1^2x^2+(\tfrac{a_{12}\alpha_2}{4a_{21}})^2y^2\right)-\tfrac{b_1}{2k}x^{\theta+2}-\tfrac{a_{12}^{\theta+1}}{4\sigma^{\theta}}w^{\theta+1}\\
&\quad+b_1x\left(x+\tfrac{a_{12}y}{4a_{21}}+\tfrac{a_{12}u}{2\sigma}+\tfrac{a_{12}w}{\sigma}\right)^{\theta}-\tfrac{a_{12}^{\theta+1}}{2^{\theta+3}\sigma^{\theta}}u^{\theta+1}-\tfrac{a_{12}^{\theta+1}b_2}{2\times4^{\theta+1}a_{21}^{\theta+1}}y^{\theta+1}\\
&\le \tfrac{\theta}{2}\left(x+\tfrac{a_{12}y}{4a_{21}}+\tfrac{a_{12}u}{2\sigma}+\tfrac{a_{12}w}{\sigma}\right)^{\theta-1}\left(\alpha_1x+\left(\tfrac{a_{12}\alpha_2}{4a_{21}}\right)y+\left(\tfrac{a_{12}}{2\sigma}\right)u+\left(\tfrac{a_{12}}{\sigma}\right)w\right)^2-\tfrac{b_1}{2k}x^{\theta+2}\\
&\quad-\tfrac{a_{12}^{\theta+1}}{4\sigma^{\theta}}w^{\theta+1}+b_1x\left(x+\tfrac{a_{12}y}{4a_{21}}+\tfrac{a_{12}u}{2\sigma}+\tfrac{a_{12}w}{\sigma}\right)^{\theta}-\tfrac{a_{12}^{\theta+1}}{2^{\theta+3}\sigma^{\theta}}u^{\theta+1}-\tfrac{a_{12}^{\theta+1}b_2}{2\times4^{\theta+1}a_{21}^{\theta+1}}y^{\theta+1}\\
&\le \tfrac{\theta4^{\theta}}{2}\left(x^{\theta+1}+\tfrac{a_{12}^{1+\theta}}{(4a_{21})^{\theta+1}}y^{\theta+1}+\tfrac{a_{12}^{1+\theta}}{(2\sigma)^{1+\theta}}u^{\theta+1}+\tfrac{a_{12}^{1+\theta}}{\sigma^{1+\theta}}w^{\theta+1}\right)\alpha_{max}^2-\tfrac{b_1}{2k}x^{\theta+2}-\tfrac{a_{12}^{\theta+1}}{4\sigma^{\theta}}w^{\theta+1}\\
&\quad+b_1x\left(x+\tfrac{a_{12}y}{4a_{21}}+\tfrac{a_{12}u}{2\sigma}+\tfrac{a_{12}w}{\sigma}\right)^{\theta}-\tfrac{a_{12}^{\theta+1}}{2^{\theta+3}\sigma^{\theta}}u^{\theta+1}-\tfrac{a_{12}^{\theta+1}b_2}{2\times4^{\theta+1}a_{21}^{\theta+1}}y^{\theta+1}\\
&\le \max_{x\in(0,+\infty)}\left\{\frac{4^{\theta}\theta\alpha_{max}^2}{2}x^{\theta+1}-\frac{b_1}{2k}x^{\theta+2}+b_1x\left(x+\frac{a_{12}y}{4a_{21}}+\frac{a_{12}u}{2\sigma}+\frac{a_{12}w}{\sigma}\right)^{\theta}\right\}\\
&\quad-\tfrac{w^{\theta+1}}{2\sigma^{\theta}}\left(\tfrac{a_{12}^{\theta+1}}{2}-\tfrac{\theta4^{\theta}a_{12}^{\theta+1}\alpha_{max}^2}{\sigma}\right)-\tfrac{u^{\theta+1}}{2(2\sigma)^{\theta}}\left(\tfrac{a_{12}^{\theta+1}}{4}-\theta4^{\theta}a_{12}^{1+\theta}\alpha_{max}^2\right)\\
&\quad-\tfrac{y^{\theta+1}}{2(4^{\theta}a_{21}^{\theta+1})}\left(a_{12}^{\theta+1}b_2-\theta4^{\theta}a_{12}^{1+\theta}\alpha_{max}^2\right)\\
&\le \max_{x\in(0,+\infty)}\left\{\frac{4^{\theta}\theta\alpha_{max}^2}{2}x^{\theta+1}-\frac{b_1}{2k}x^{\theta+2}+b_1x\left(x+\frac{a_{12}y}{4a_{21}}+\frac{a_{12}u}{2\sigma}+\frac{a_{12}w}{\sigma}\right)^{\theta}\right\},
\end{aligned}$$

where $\alpha_{max}^2=\max\{\alpha_1^2,\alpha_2^4\}$, holding $4^{\theta}\theta\le\min\frac{1}{\alpha_{max}^2}\{\frac{1}{4},\frac{\sigma}{2},b_2\}$. It is clear that B has an upper bound.

Define a Lyapunov function as follows

$$\widetilde{V}_4(x,y,u,w) = M\widetilde{V}_2 + \widetilde{V}_3 - \ln w - \ln u,$$

where $M > 0$ is a fixed constant satisfying $-M\lambda_0 + 2\sigma + \tilde{B} < -2$. Moreover $\widetilde{V}_4(x,y,u,w)$ is not only continuous, but also tends to $+\infty$ as (x,y,u,w) approaches the boundary of $\mathbb{R}^4_+$. Hence it must be lower bounded and achieve this lower bound at a point (x^0,y^0,u^0,w^0) in the interior of $\mathbb{R}^4_+$. Then we denote the C^2-function as

$$V^*(x,y,u,w) = \widetilde{V}_4(x,y,u,w) - \widetilde{V}_4(x^0,y^0,u^0,w^0).$$

Applying Itô formula, we have

$$L(-\ln w - \ln u) = 2\sigma - \frac{\sigma w}{u} - \frac{\sigma xy}{w(m_1+m_2x+m_3y)}. \tag{3.6}$$

Therefore, by (3.3)-(3.6), one can get that

$$\begin{aligned} LV^* \le{}& -M\lambda_0 - \tfrac{b_1}{2k}x^{\theta+2} - \tfrac{a_{12}^{\theta+1}b_2}{2\times 4^{\theta+1}a_{21}^{\theta+1}}y^{\theta+1} - \tfrac{a_{12}^{\theta+1}}{4\sigma^\theta}w^{\theta+1} - \tfrac{a_{12}^{\theta+1}}{2^{\theta+3}\sigma^\theta}u^{\theta+1} \\ &+ \tfrac{Ma_{21}}{b_2}\left(m_3 + 4\sqrt[4]{a_{21}c_1c_2\sigma^2k}\tfrac{a_{12}}{b_1m_1} - \tfrac{\sigma b_2}{wMa_{21}}\right)\tfrac{xy}{m_1+m_2x+m_3y} + \widetilde{B} + 2\sigma, \end{aligned}$$

where $\widetilde{B} = \sup B$. We denote $\frac{Ma_{21}}{b_2}(m_3 + 4\sqrt[4]{a_{21}c_1c_2\sigma^2k}\frac{a_{12}}{b_1m_1}) = \widetilde{\lambda}$ for convenience.

We consider the bounded closed set

$$U_\epsilon = \{(x,y,u,w)\in\mathbb{R}^4_+ | \epsilon < x < \frac{1}{\epsilon}, \epsilon < y < \frac{1}{\epsilon}, \epsilon < u < \frac{1}{\epsilon^3}, \epsilon < w < \frac{1}{\epsilon^2}\},$$

where $0 < \epsilon < 1$ is a sufficiently small number. In the set $\mathbb{R}^4_+ \setminus U_\epsilon$, we can choose ϵ sufficiently small such that the following conditions hold:

$$\begin{aligned} &A_1: \epsilon \le \frac{m_1(\theta+1)}{\widetilde{\lambda}\theta}, \qquad \frac{\widetilde{\lambda}\epsilon}{m_1(\theta+1)} \le \frac{a_{12}^{\theta+1}b_2}{2\times 4^{\theta+1}a_{21}^{\theta+1}}, \\ &A_2: \epsilon \le \frac{(m_1+m_3\epsilon)(\theta+2)}{\widetilde{\lambda}(\theta+1)}, \quad \frac{\widetilde{\lambda}\epsilon}{(m_1+m_3\epsilon)(\theta+2)} \le \frac{b_1}{2k}, \\ &A_3: \frac{-\sigma}{\epsilon(m_1+m_2\epsilon+m_3\epsilon)} + C \le -1, \quad A_4: \frac{-\sigma}{\epsilon} + C \le -1, \\ &A_5: -\frac{b_1}{4k\epsilon^{\theta+2}} + C \le 1, \qquad A_6: -\frac{a_{12}^{\theta+1}b_2}{4^{\theta+2}a_{21}^{\theta+2}\epsilon^{\theta+1}} + C \le -1, \\ &A_7: -\frac{a_{12}^{\theta+1}}{4\sigma^\theta(\epsilon^3)^{\theta+1}} + C \le -1, \qquad A_8: -\frac{a_{12}^{\theta+1}}{2^{\theta+3}\sigma^\theta(\epsilon^2)^{\theta+1}} + C \le -1. \end{aligned}$$

Then we divide $\mathbb{R}^4_+ \setminus U_\epsilon$ into eight domains as if

$$\begin{aligned} D_1 =& \{(x,y,u,w)\in\mathbb{R}^4_+ : 0 < x < \epsilon\}, \\ D_2 =& \{(x,y,u,w)\in\mathbb{R}^4_+ : 0 < y < \epsilon\}, \\ D_3 =& \{(x,y,u,w)\in\mathbb{R}^4_+ : 0 < w < \epsilon^3, x > \epsilon, y > \epsilon\}, \\ D_4 =& \{(x,y,u,w)\in\mathbb{R}^4_+ : 0 < u < \epsilon^2, w > \epsilon\}, \end{aligned}$$

$$\begin{aligned}
D_5 =&\{(x,y,u,w)\in \mathbb{R}^4_+ : x > \frac{1}{\epsilon}\},\\
D_6 =&\{(x,y,u,w)\in \mathbb{R}^4_+ : y > \frac{1}{\epsilon}\},\\
D_7 =&\{(x,y,u,w)\in \mathbb{R}^4_+ : u > \frac{1}{\epsilon^3}\},\\
D_8 =&\{(x,y,u,w)\in \mathbb{R}^4_+ : w > \frac{1}{\epsilon^2}\}.
\end{aligned}$$

Clearly, $D_\epsilon^C = \bigcup_{i=1}^{8} D_\epsilon^i$. Next, we will show that $LV^*(x,y,u,w) \le -1$ on $U_\epsilon^C \times \mathbb{R}_+$, which is equivalent to show it on the above eight domains.

Case 1. If $(x,y,u,w)\in D_1$, one can see that

$$\frac{xy}{m_1+m_2x+m_3y} \le \frac{\epsilon y}{m_1} \le \frac{\epsilon}{m_1}\frac{\theta+y^{\theta+1}}{\theta+1} = \frac{\epsilon\theta}{m_1(\theta+1)} + \frac{\epsilon y^{\theta+1}}{m_1(\theta+1)}.$$

Thus,

$$\begin{aligned}
LV^* &\le -M\lambda_0 + \widetilde{\lambda}\left(\frac{\epsilon\theta}{m_1(\theta+1)} + \frac{\epsilon y^{\theta+1}}{m_1(\theta+1)}\right) + \widetilde{B} + 2\sigma - \frac{a_{12}^{\theta+1}b_2}{2\times 4^{\theta+1}a_{21}^{\theta+1}}y^{\theta+1}\\
&= -M\lambda_0 + \frac{\widetilde{\lambda}\epsilon\theta}{m_1(\theta+1)} + \left(\frac{\widetilde{\lambda}\epsilon}{m_1(\theta+1)} - \frac{a_{12}^{\theta+1}b_2}{2\times 4^{\theta+1}a_{21}^{\theta+1}}\right)y^{\theta+1} + \widetilde{B} + 2\sigma\\
&\le -2+1=-1,
\end{aligned}$$

by virtue of the condition A_1. Thus, $LV^* \le -1$ on D_1.

Case 2. If $(x,y,u,w)\in D_2$, one can see that

$$\frac{xy}{m_1+m_2x+m_3y} \le \frac{\epsilon}{m_1+m_3\epsilon}\frac{(\theta+1)+x^{\theta+2}}{\theta+2} = \frac{\epsilon(\theta+1)}{(m_1+m_3\epsilon)(\theta+2)} + \frac{\epsilon x^{\theta+2}}{(m_1+m_3\epsilon)(\theta+2)}.$$

Hence,

$$\begin{aligned}
LV^* &\le -M\lambda_0 + \widetilde{\lambda}\left(\frac{\epsilon(\theta+1)}{(m_1+m_3\epsilon)(\theta+2)} + \frac{\epsilon x^{\theta+2}}{(m_1+m_3\epsilon)(\theta+2)}\right) + \widetilde{B} + 2\sigma - \frac{b_1}{2k}x^{\theta+2}\\
&= -M\lambda_0 + \frac{\widetilde{\lambda}\epsilon(\theta+1)}{(m_1+m_3\epsilon)(\theta+2)} + \left(\frac{\widetilde{\lambda}\epsilon}{(m_1+m_3\epsilon)(\theta+2)} - \frac{b_1}{2k}\right)x^{\theta+2} + \widetilde{B} + 2\sigma\\
&\le -2+1=-1,
\end{aligned}$$

by the condition A_2, we can get $LV^* \le -1$ on D_2.

Case 3. If $(x,y,u,w)\in D_3$, it yields

$$\begin{aligned}
LV^* &\le \frac{-\sigma xy}{w(m_1+m_2x+m_3y)} + C - \frac{b_1}{4k}x^{\theta+2} - \frac{a_{12}^{\theta+1}b_2}{4^{\theta+2}a_{21}^{\theta+2}}y^{\theta+1}\\
&\le \frac{-\sigma}{\epsilon(m_1+m_2\epsilon+m_3\epsilon)} + C,
\end{aligned}$$

where $C = \sup\{-\frac{b_1}{4k}x^{\theta+2} - \frac{a_{12}^{\theta+1}b_2}{4^{\theta+2}a_{21}^{\theta+2}}y^{\theta+1} + \frac{\widetilde{\lambda}xy}{m_1+m_2x+m_3y} + \widetilde{B} + 2\sigma\}$. By the condition A_3, we can get $LV^* \le -1$ on D_3.

Case 4. If $(x,y,u,w)\in D_4$, it yields

$$\begin{aligned}
LV^* &\le \frac{-\sigma w}{u} + C - \frac{b_1}{4k}x^{\theta+2} - \frac{a_{12}^{\theta+1}b_2}{4^{\theta+2}a_{21}^{\theta+2}}y^{\theta+1}\\
&\le \frac{-\sigma}{\epsilon} + C,
\end{aligned}$$

which, together with the condition A_4, induces that $LV^* \leq -1$ on D_4.

Case 5. If $(x, y, u, w) \in D_5$, we can derive

$$\begin{aligned} LV^* &\leq \tfrac{-\sigma xy}{w(m_1+m_2x+m_3y)} + C - \tfrac{b_1}{4k}x^{\theta+2} - \tfrac{a_{12}^{\theta+1}b_2}{4^{\theta+2}a_{21}^{\theta+2}}y^{\theta+1} \\ &\leq -\frac{b_1}{4k\epsilon^{\theta+2}} + C. \end{aligned}$$

According to the condition A_5, we can deduce that $LV^* \leq -1$ on D_5.

Case 6. If $(x, y, u, w) \in D_6$, it yields

$$\begin{aligned} LV^* &\leq \tfrac{-\sigma xy}{w(m_1+m_2x+m_3y)} + C - \tfrac{b_1}{4k}x^{\theta+2} - \tfrac{a_{12}^{\theta+1}b_2}{4^{\theta+2}a_{21}^{\theta+2}}y^{\theta+1} \\ &\leq -\tfrac{a_{12}^{\theta+1}b_2}{4^{\theta+2}a_{21}^{\theta+2}\epsilon^{\theta+1}} + C, \end{aligned}$$

which follows from the condition A_6 that $LV^* \leq -1$ on D_6.

Case 7. If $(x, y, u, w) \in D_7$, we have

$$\begin{aligned} LV^* &\leq \tfrac{-\sigma xy}{w(m_1+m_2x+m_3y)} + C - \tfrac{\sigma w}{u} - \tfrac{a_{12}^{\theta+1}}{4\sigma^\theta}w^{\theta+1} \\ &\leq -\tfrac{a_{12}^{\theta+1}}{4\sigma^\theta(\epsilon^3)^{\theta+1}} + C. \end{aligned}$$

By the condition A_7, we can conclude that $LV^* \leq -1$ on D_7.

Case 8. If $(x, y, u, w) \in D_8$, one can derive that

$$\begin{aligned} LV^* &\leq \tfrac{-\sigma xy}{w(m_1+m_2x+m_3y)} + C - \tfrac{\sigma w}{u} - \tfrac{a_{12}^{\theta+1}}{4\sigma^\theta}w^{\theta+1} \\ &\leq -\tfrac{a_{12}^{\theta+1}}{2^{\theta+3}\sigma^\theta(\epsilon^2)^{\theta+1}} + C. \end{aligned}$$

By the condition A_8, we can conclude that $LV^* \leq -1$ on D_8.

So we get that

$$LV^*(x, y, u, w) \leq -1 \text{ for any } (x, y, u, w) \in \mathbb{R}_+^4 \setminus U_\epsilon.$$

That is, the conditions in Lemma 3.2 hold. Hence, we obtain that system (2.1) has a solution which is a stationary Markov process. This completes the proof.

4. Main results in the weak kernel case

For the weak kernel $K(t) = \sigma e^{-\sigma t}$, let

$$u(t) = \int_{-\infty}^t K(t-s)\frac{xy}{m_1 + m_2x + m_3y}ds.$$

Then system (2.1) becomes the following equivalent system:

$$\begin{cases} dx = \left(b_1x\left(1 - \frac{x}{k}\right) - \frac{a_{12}xy}{m_1+m_2x+m_3y}\right)dt + \alpha_1 x dB_1(t), \\ dy = (-b_2y + a_{21}u)dt + \alpha_2 y dB_2(t), \\ du = \sigma\left(\frac{xy}{m_1+m_2x+m_3y} - u\right)dt. \end{cases} \tag{4.1}$$

By similar and simpler method as section 3, we directly draw the following conclusions:

Theorem 4.1. *For any given initial value $(x(0), y(0), u(0)) \in \mathbb{R}^3_+$, there exists a unique positive solution $(x(t), y(t), u(t))$ of system (4.1) on $t \geq 0$ and the solution will remain in $\mathbb{R}^3_+$ with probability one.*

Theorem 4.2. *Let $(x(t), y(t), u(t))$ be a solution of system (4.1) with any initial value $(x(0), y(0), u(0)) \in \mathbb{R}^3_+$. Then*
(1) If $b_1 < \frac{\alpha_1^2}{2}$, then $\lim_{t\to\infty} x(t) = 0$ a.s.,
(2) If $b_1 > \frac{\alpha_1^2}{2}$, then for almost all $\omega \in \Omega$, we have

$$\limsup_{t\to\infty} \frac{1}{t} \ln \left(\sqrt{\frac{k}{a_{21} b_2 (m_1 + m_2 k)}} y(t) + \frac{1}{\sigma} u(t) \right) \leq \mu \, a.s.,$$

where $\mu = \min\{b_2, \sigma\}(\sqrt{R_0} - 1) I_{\{R_0 \leq 1\}} + \max\{b_2, \sigma\}(\sqrt{R_0} - 1) I_{\{R_0 \geq 1\}} + \alpha_1 b_2 \sqrt{\frac{R_0}{2b_1}}$ and $R_0 = \frac{a_{21} k}{b_2(m_1 + m_2 k)}$. Especially, if $\mu < 0$, then the predator population y will die out exponentially with probability one, i,e.,

$$\lim_{t\to\infty} y(t) = 0, \quad a.s.$$

Moreover, the distribution of $x(t)$ converges weakly to the measure which has the density

$$\pi(u) = Q \alpha_1^{-2} u^{-2 + \frac{2b_1}{\alpha_1^2}} e^{-\frac{2b_1}{k\alpha_1^2}}, u \in (0, \infty),$$

where $Q = [\alpha_1^{-2} (\frac{k\alpha_1^2}{2b_1})^{-1+\frac{2b_1}{\alpha_1^2}} \Gamma(\frac{2b_1}{\alpha_1^2} - 1)]^{-1}$ satisfying $\int_0^\infty \pi(u) du = 1$.

Theorem 4.3. *Assume that $b_1 > \frac{\alpha_1^2}{2}, b_2 > \frac{\alpha_2^2}{2}$, and $R_0^S = \frac{a_{21} k (1 - \frac{\alpha_1^2}{2b_1})^3}{(m_1 + m_2 k)(b_2 + \frac{\alpha_2^2}{2})} > 1$. Then system (4.1) exists a solution $P^*(t) \triangleq (x^*(t), y^*(t), u^*(t))$, which is a stationary Markov process.*

5. Numerical simulations

In this section, we numerically simulate the solutions of system (2.1) with strong kernel, which illustrate the different dynamics between the deterministic model (1.2) and the stochastic model (2.1) by Milstiein Higher Order Method [35].
Case 1. We choose a set of parameters:

$$b_1 = 0.4, b_2 = 0.2, a_{12} = 0.1, a_{21} = 0.2, k = 0.6, \sigma = 0.8, m_1 = 0.1, m_2 = 0.5, m_3 = 0.3.$$

Obviously, the condition $b_2 < \frac{a_{21} k}{m_1 + k m_2}$ holds. Hence, the deterministic system (1.2) with the strong kernel has a positive equilibrium $(x_*, y_*) = (0.4589, 0.4316)$, which is globally asymptotically stable, illustrated in the left graph of Figure 1. Then we consider the effect of the white noises on the system. There we choose $\alpha_1 = 0.05, \alpha_2 = 0.02$ satisfying $b_1 > \frac{\alpha_1^2}{2}$. By Theorem 2.2, we know that the prey will be persistence in mean. The picture also shows that if white noise is small

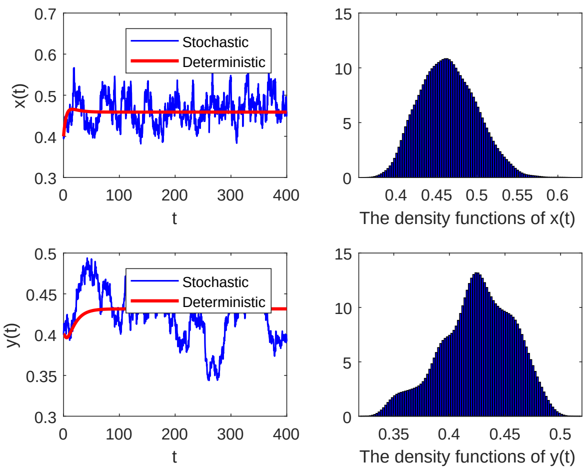

Figure 1. Left: The solutions of the stochastic system (2.1) and deterministic system (1.2) with strong kernel; Right: The density function diagrams of $x(t), y(t)$ with the initial value $(x(t), y(t)) = (0.4, 0.4), t \in (-\infty, 0]$.

the predator is also persistence in mean. The results indicate that the stochastic turbulence has no obvious influence when the white noise is relatively small.

Case 2. We choose the same parameters $b_1, b_2, a_{12}, a_{21}, k, \sigma, m_1, m_2, m_3$ as Case 1. That is, the positive equilibrium has the same value as Figure 1, illustrated in Figure 2. However, when we increase the intensities of white noise as $\alpha_1 = 0.9$, such that $b_1 < \frac{\alpha_1^2}{2}$, by Theorem 2.2, we know that the prey will tend to extinction. Moreover the predator will extinct as prey disappears. The right graph of the Figure 2 implies that prey-predator species will tend to extinction with the stronger white noise. Furthermore, if we increase the intensities of the white noises of the prey and predator, such as $\alpha_1 = 0.02, \alpha_2 = 0.5$, by Theorem 2.2, we know that the predator will tend to extinct, but the prey still tend to persistence in the mean, which are illustrated in Figure 3.

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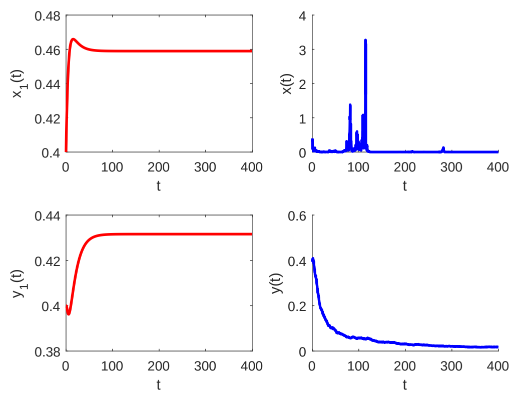

Figure 2. Left: The solutions of the deterministic system (1.2) with strong kernel; Right: The solutions of the stochastic system (2.1), where $\alpha_1 = 0.9, \alpha_2 = 0.02$ with the initial value $(x(t), y(t)) = (0.4, 0.4), t \in (-\infty, 0]$

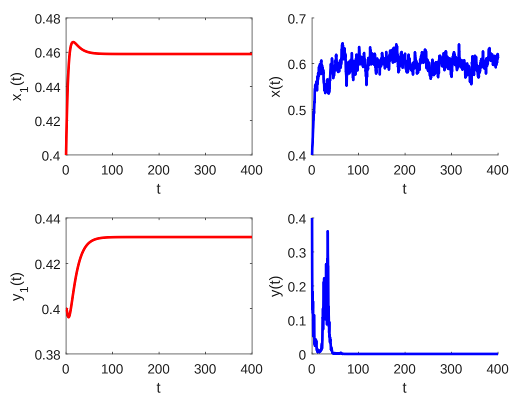

Figure 3. Left: The solutions of the deterministic system (1.2) with strong kernel; Right: The solutions of the stochastic system (2.1), where $\alpha_1 = 0.02, \alpha_2 = 0.5$ with the initial value $(x(t), y(t)) = (0.4, 0.4), t \in (-\infty, 0]$.

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