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New Peakons and Periodic Peakons of the Modified Camassa-Holm Equation*

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Abstract In this paper, we obtain new peakon and periodic peakon solutions to a modified Camassa-Holm equation. We change the modified Camassa-Holm equation into a planar system. Then the first integral and algebraic curves of this system are obtained. By using the first integral and algebraic curves, a new peakon solution is given by hyperbolic function. Moreover, some new periodic peakons are given by elliptic functions and triangle functions.

Keywords Camassa-Holm equation, Peakon, Periodic peakon, Solitary wave, Periodic wave.

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1. Introduction

In recent years, studies on Camassa-Holm equations have received considerable attention because these equations have many applications in physics. The Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1.1}$$

was proposed by Camassa and Holm [1] as a model equations for shallow water unidirectional nonlinear dispersion waves, where $u(x,t)$ representing the waters free surface over a flat bed. Equation (1.1) admits the peakons and periodic peakons in the following forms [1,5]

$$u(x,t) = ce^{-|x-ct|}$$

and

$$u(x,t) = \frac{c}{\sinh(1/2)} \cosh\left(\frac{1}{2} - (x-ct) + [x-ct]\right),$$

where the notation $[x]$ denotes the largest integer part of the real number $x \in \mathbb{R}$. Lenells [4] obtained smooth solitary wave solutions to the famous Korteweg-de Vries equation

$$u_t - 6uu_x + uu_{xxx} = 0, \tag{1.2}$$

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and showed that the smooth traveling waves of (1.1) naturally correspond to traveling waves of (1.2).

The μ-Camassa-Holm (μCH) equation

$$\mu(u_t) - u_{xxt} = -2\mu(u)u_x + 2u_x u_{xx} + u u_{xxx} \tag{1.3}$$

was introduced as an integrable equation arising in the study of the diffeomorphism group of the circle, where $u(x,t)$ is a real-valued spatially periodic function and $\mu(u) = \int_{S^1} u(x,t)dx$ denotes its mean. Lenells et al. [6] obtained that equation (1.3) admits periodic peakons

$$u(x,t) = \frac{c}{26}\left(12(x-ct)^2 + 23\right),$$

where $|x-ct| \le \frac{1}{2}$ and $u(x,t)$ is extended periodically to the real line.

In this paper, we consider the modified Camassa-Holm (mCH) equation

$$u_t - u_{xxt} = u u_{xxx} + 2u_x u_{xx} - 3u^2 u_x, \tag{1.4}$$

where $x \in \mathbb{R}$ and $t > 0$. Wazwaz [12] obtained some solitary wave solutions to (1.4) by using the extended tanh method and the rational hyperbolic functions method. Moreover, based on the method of complete discrimination system for polynomial, Deng [2] obtained some exact travelling wave solutions to (1.4). Since the nonlinear partial differential equations have various traveling wave solutions [3,7–11], inspired by the above, the aim of this paper is to construct new peakons and periodic peakons solutions by using first integral and algebraic curves. A peakon solution is given by a hyperbolic function, and some new periodic peakons are given by elliptic functions and triangle functions.

The rest of the paper is organized as follows. In Section 2, we change (1.4) into a planar system. Then we obtain the first integral of the planar system, and use Maple to draw the bifurcation of each algebraic curve on the phase plane. In Section 3, we obtain new peakons and periodic peakons solutions to equation (1.4).

2. First integral and algebraic curve

By substituting $u(x,t) = \phi(\xi)$ with $\xi = x - ct$ into equation (1.4), it follows that

$$-c\phi' + c\phi''' = \phi\phi''' + 2\phi'\phi'' - 3\phi^2\phi', \tag{2.1}$$

where ϕ' is the derivative with respect to ξ. Integrating equation (2.1) once, we obtain

$$(\phi - c)\,\phi'' + \frac{1}{2}(\phi')^2 - \phi^3 + c\phi = g, \tag{2.2}$$

where g is the integral constant.

Letting $y = \frac{d\phi}{d\xi}$, then we obtain the following planar dynamic system

$$\begin{cases} \dfrac{d\phi}{d\xi} = y, \\ \dfrac{dy}{d\xi} = \dfrac{-\frac{1}{2}y^2 + \phi^3 - c\phi + g}{\phi - c}. \end{cases} \tag{2.3}$$

Obviously, system (2.3) has the first integral

$$H(\phi, y) = (\phi - c)\left[y^2 - \frac{1}{2}\left(\phi^3 + d_0\phi^2 + d_1\phi + d_2\right)\right] = h, \tag{2.4}$$

where $d_2 = c^3 - 2c^2 + 4g$, $d_1 = c^2 - 2c$ and $d_0 = c$, and h is an integral constant. Note that system (2.3) is discontinuous on the singular line $\phi = c$. Let $F(\phi) = \phi^3 + d_0\phi^2 + d_1\phi + d_2$, then (2.4) can be rewritten as

$$H(\phi, y) = (\phi - c)\left[y^2 - \frac{1}{2}F(\phi)\right] = h.$$

We only consider the case $h = 0$. we assume that $d_2 = c^3 - 2c^2 + 4g = 0$, which yields $g = \frac{2c^2 - c^3}{4}$. For $y = 0$, we can get the roots of the equation $H(\phi, 0) = 0$, if the roots of the equation $F(\phi) = 0$ is found. Obviously, $\phi_0 = 0$ is a root of the equation $F(\phi) = 0$. Define $\Delta = -3c^2 + 8c$. When $\Delta > 0$, there are two real roots $\phi_1 = \frac{-c+\sqrt{\Delta}}{2}$ and $\phi_2 = \frac{-c-\sqrt{\Delta}}{2}$ for the equation $F(\phi) = 0$. When $\Delta = 0$, there is a double root $\phi_3 = -\frac{c}{2}$. When $\Delta < 0$, there exists conjugate imaginary roots $\phi_4 = \frac{-c+\sqrt{-\Delta}\ i}{2}$ and $\phi_5 = \frac{-c-\sqrt{-\Delta}\ i}{2}$, where $i^2 = -1$. The graphs of the algebraic curve $H(\phi, y) = 0$ are presented in the following Proposition. (For simplicity, we only consider the case $c > 0$.)

Proposition 2.1. *Let $g = \frac{2c^2 - c^3}{4}$, then we have the following conclusions.*

1. For $0 < c < \frac{2}{3}$, the algebraic curves $H(\phi, y) = 0$ consist of a closed curve and a open curve (see Fig. 1(a)), the closed curve corresponds a smooth periodic wave solution.

2. For $\frac{2}{3} < c < 2$, the algebraic curves $H(\phi, y) = 0$ consists of a closed orbit and a open curve (see Fig. 1(b)), which corresponds smooth periodic wave solution and periodic peakon solution, respectively.

3. For $c = 2$, the algebraic curves $H(\phi, y) = 0$ consists of a homoclinic orbit and two heteroclinic orbits (see Fig. 1(c)), which corresponds smooth solitary wave solution and periodic peakon solution, respectively.

4. For $2 < c < \frac{8}{3}$, the algebraic curves $H(\phi, y) = 0$ consists of a closed orbit and a open curve (see Fig. 1(d)), which corresponds smooth periodic wave solution and periodic peakon solution, respectively.

5. For $c \geq \frac{8}{3}$, the algebraic curves $H(\phi, y) = 0$ is a open curve (see Fig. 1(e) and Fig. 1(f)), which corresponds a periodic peakon solution.

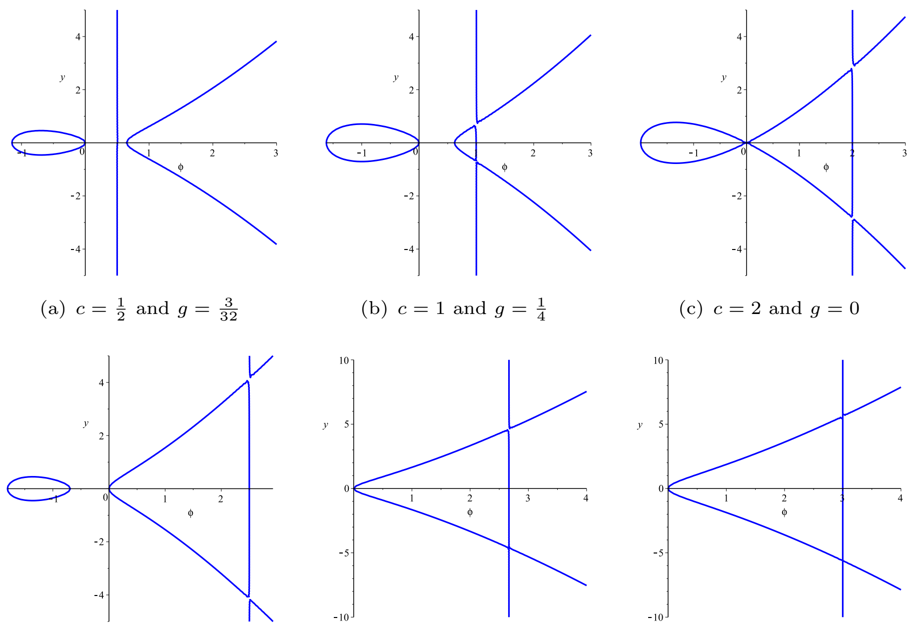
3. Peakon and periodic peakon solutions

In this Section, from Proposition 2.1, we can obtain peakon and periodic peakon solutions to equation (1.4).

Type 1: Peakon

When $c = 2$, by (2.4), we obtain an algebraic curve Γ_1:

$$y = \pm\frac{1}{\sqrt{2}}\ (\phi - \phi_0)\sqrt{\phi - \phi_2}. \tag{3.1}$$



(a) $c = \frac{1}{2}$ and $g = \frac{3}{32}$ (b) $c = 1$ and $g = \frac{1}{4}$ (c) $c = 2$ and $g = 0$

(d) $c = \frac{5}{2}$ and $g = -\frac{25}{32}$ (e) $c = \frac{8}{3}$ and $g = -\frac{32}{27}$ (f) $c = 3$ and $g = -\frac{9}{4}$

Figure 1. (Color online.) The graphs of the algebraic curve $H(\phi, y) = 0$.

where $\phi_0 = 0$ and $\phi_2 = -2$. By substituting (3.1) into (2.3) and integrating the first expression of the system (2.3), we can get

$$\pm\sqrt{2}\int_2^\phi \frac{d\phi}{\phi\sqrt{\phi+2}} = \int_0^\xi d\xi,$$

$$\pm 2\coth^{-1}\sqrt{\frac{\phi+2}{2}}\Bigg|_2^\phi = \xi.$$

By using identity $1 - \coth^2(x) = -\text{csch}^2(x)$, we obtain that the peakon solution to equation (1.4) can be written as

$$\phi(\xi) = 2\text{csch}^2\left(A + \frac{1}{2}|\xi|\right), \tag{3.2}$$

where $A = \coth^{-1}\left(\sqrt{2}\,\right)$ and $\xi = x - 2t$. The profile of peakon is shown in Fig. 2.

Remark 3.1. If $c = 2$, as shown in Fig. 1(c), we obtain that there is a homoclinic orbit which corresponds to a smooth solitary wave solution. By (2.4), we also obtain another algebraic curve Γ_2:

$$y = \pm\frac{1}{\sqrt{2}}\,(\phi_0 - \phi)\sqrt{\phi - \phi_2}. \tag{3.3}$$

By substituting (3.3) into (2.3) and integrating the first expression of the system (2.3), we have

$$\pm\sqrt{2}\int_{-2}^\phi \frac{d\phi}{\phi\sqrt{\phi+2}} = \int_0^\xi d\xi,$$

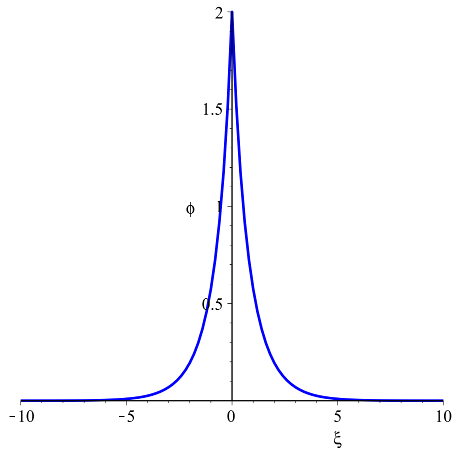


Figure 2. (Color online.) The profile of the peakon (3.2).

$$\pm 2\tanh^{-1}\sqrt{\frac{\phi+2}{2}}\Bigg|_{-2}^{\phi} = \xi.$$

By using identity $1-\tanh^2(x) = \mathrm{sech}^2(x)$, the smooth solitary wave solutions to equation (1.4) can be expressed as

$$\phi(\xi) = -2\mathrm{sech}^2\left(\frac{1}{2}\xi\right), \tag{3.4}$$

where $\xi = x - 2t$. The profile of smooth solitary wave is shown in Fig. 3.

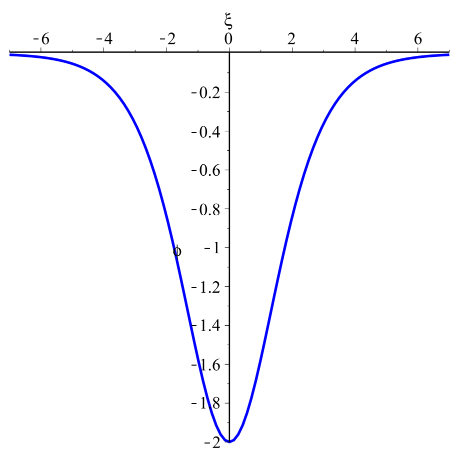


Figure 3. (Color online.) The profile of the smooth solitary wave (3.4).

Type 2: Periodic peakon

From Proposition 2.1, it is easy to see that there are periodic peakons, if $\frac{2}{3} < c < 2$ or $c > 2$. In the following, we will discuss these cases separately.

Case 1. When $\frac{2}{3} < c < 2$, for simplicity, we take $c = 1$, by (2.4), we obtain an algebraic curve Γ_3:

$$y = \pm\frac{1}{\sqrt{2}}\ \sqrt{(\phi-\phi_0)(\phi-\phi_1)(\phi-\phi_2)}. \tag{3.5}$$

By substituting (3.5) into (2.3) and integrating the first expression of the system (2.3), we obtain that the traveling wave solution to equation (1.4) can be written as

$$\phi(\xi) = \frac{\phi_1}{\mathrm{cn}^2\Big(\sqrt{\frac{\sqrt{5}}{8}}\ \xi, k_1\Big)}, \quad -T_1 \le \xi \le T_1, \tag{3.6}$$

where $k_1 = \sqrt{\frac{\phi_2}{\phi_2-\phi_1}}$ and $\xi = x - t$. By extending the formula (3.6) to the entire real axis, we have the following periodic peakon

$$\phi(\xi) = \frac{\phi_1}{\mathrm{cn}^2\Big(\sqrt{\frac{\sqrt{5}}{8}}\ (\xi - 2nT_1), k_1\Big)}, \quad (2n-1)\ T_1 \le \xi \le (2n+1)\ T_1, \tag{3.7}$$

where $n \in \mathbb{Z}^+$ and $T_1 = \sqrt{\frac{8}{\sqrt{5}}}\ \mathrm{cn}^{-1}\left(\sqrt{\frac{\sqrt{5}-1}{2}}, k_1\right)$. The profile of periodic peakon is shown in Fig. 4.

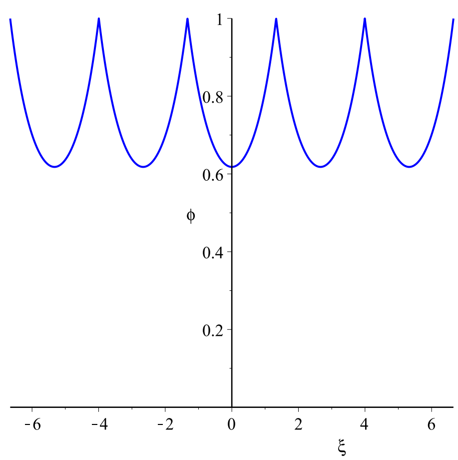


Figure 4. (Color online.) The profile of the periodic peakon (3.7).

Remark 3.2. If $\frac{2}{3} < c < 2$, as shown in Fig. 1(b), we obtain that there is a periodic orbit. Rewrite (2.4) as an algebraic curve Γ_4:

$$y = \pm\frac{1}{\sqrt{2}}\ \sqrt{(\phi_0-\phi)(\phi_1-\phi)(\phi-\phi_2)}. \tag{3.8}$$

By substituting (3.8) into (2.3) and integrating the first expression of the system (2.3), we obtain that equation (1.4) has smooth periodic wave solutions in the form

$$\phi(\xi) = \phi_2 \mathrm{cn}^2\left(\sqrt{\frac{\phi_1-\phi_2}{8}}\xi, k_1\right), \tag{3.9}$$

where $\xi = x - t$. The profile of smooth periodic wave is shown in Fig. 5.

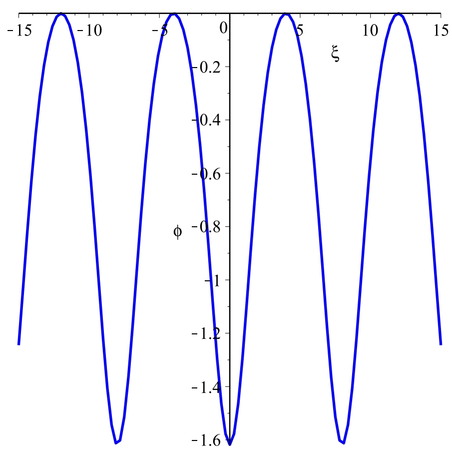


Figure 5. (Color online.) The profile of the smooth periodic wave (3.9).

Case 2. When $c = \frac{8}{3}$, by (2.4), we obtain an algebraic curve Γ_5:

$$y = \pm\frac{1}{\sqrt{2}}\,(\phi-\phi_3)\sqrt{\phi-\phi_0}. \tag{3.10}$$

The profile of periodic peakon is shown in Fig. 7.

By substituting (3.10) into (2.3) and integrating the first expression of the system (2.3), we obtain that the traveling wave solution to equation (1.4) can be written as

$$\phi(\xi) = \frac{4}{3}\tan^2\left(b_1 - \frac{1}{\sqrt{6}}\,|\xi|\right), \quad -2\sqrt{6}\,b_1 \le \xi \le 2\sqrt{6}\,b_1, \tag{3.11}$$

where $b_1 = \arctan\sqrt{2}$ and $\xi = x - \frac{8}{3}t$. Let $T_2 = 2\sqrt{6}\,b_1$. By extending the formula (3.11) to the entire real axis, we have the following periodic peakon

$$\phi(\xi) = \frac{4}{3}\tan^2\left(b_1 - \frac{1}{\sqrt{6}}\,|\xi - 2nT_2|\right), \quad (2n-1)\,T_2 \le \xi \le (2n+1)\,T_2, \tag{3.12}$$

where $n \in \mathbb{Z}^+$. The profile of periodic peakon is shown in Fig. 6.

Case 3. When $c > \frac{8}{3}$, for simplicity, we take $c = 3$, by (2.4), we obtain an algebraic curve Γ_6:

$$y = \pm\frac{1}{\sqrt{2}}\,\sqrt{(\phi-\phi_0)(\phi-\phi_4)(\phi-\phi_5)}, \tag{3.13}$$

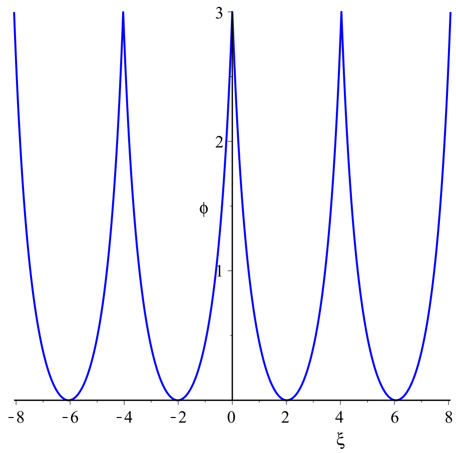


Figure 6. (Color online.) The profile of the periodic peakon (3.15).

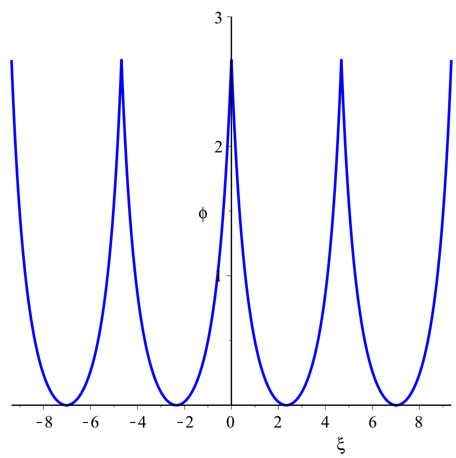


Figure 7. (Color online.) The profile of the periodic peakon (3.12).

where ϕ_4 and ϕ_5 are conjugate imaginary roots. By substituting (3.13) into (2.3) and integrating the first expression of the system (2.3), we obtain that the traveling wave solution to equation (1.4) can be written as

$$\phi(\xi) = \frac{B\left(1 + \mathrm{cn}\left(\sqrt{\frac{B}{2}}\ |\xi| + D, k_2\right)\right)}{1 - \mathrm{cn}\left(\sqrt{\frac{B}{2}}\ |\xi| + D, k_2\right)}, \quad -T_3 \le \xi \le T_3, \tag{3.14}$$

where $D = \mathrm{cn}^{-1}\left(\frac{c-B}{c+B}, k_2\right)$, $B = \sqrt{c^2 - 2c}$, $k_2 = \sqrt{\frac{2B-c}{4B}}$ and $\xi = x - 3t$. Extending the formula (3.14) to the entire real axis, we obtain the following periodic peakon

$$\phi(\xi) = \frac{B\left(1 + \mathrm{cn}\left(\sqrt{\frac{B}{2}}\ |\xi - 2nT_3| + D, k_2\right)\right)}{1 - \mathrm{cn}\left(\sqrt{\frac{B}{2}}\ |\xi - 2nT_3| + D, k_2\right)}, \quad (2n-1)\ T_3 \le \xi \le (2n+1)\ T_3, \tag{3.15}$$

where $n \in \mathbb{Z}^+$ and $T_3 = \frac{2\sqrt{2}\ (2K-D)}{\sqrt{B}}$, K is the complete elliptic integral of the first kind.

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