

Bifurcations of Double Homoclinic Loops with Inclination Flip and Nonresonant Eigenvalues

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Abstract In this work, bifurcation analysis near double homoclinic loops with W^s inclination flip of Γ_1 and nonresonant eigenvalues is presented in a four-dimensional system. We establish a Poincaré map by constructing local active coordinates approach in some tubular neighborhood of unperturbed double homoclinic loops. Through studying the bifurcation equations, we obtain the condition that the original double homoclinic loops are persistent, and get the existence or the nonexistence regions of the large 1-homoclinic orbit and the large 1-periodic orbit. At last, an analytical example is given to illustrate our main results.

Keywords Double homoclinic loops, Nonresonant eigenvalues, Inclination flip, Periodic orbit, Bifurcation.

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1. Introduction

During the last few decades, bifurcations of homoclinic or heteroclinic orbits are always widely met in applications, and they have been investigated extensively (see [1–36] and the further references therein). Notably, to the best of the authors' knowledge, only a few concerned the bifurcation of double homoclinic loops. Han and Bi [7] investigated the existence of homoclinic bifurcation curves and small and large limit cycles bifurcated from a double homoclinic loop under multiple parameter perturbations for general planar systems. Han and Chen [8] gave the number of limit cycles near double homoclinic loops under perturbations in planar Hamiltonian systems. Lu [18] obtained codimension 2 bifurcations of twisted double homoclinic loops in higher dimensional systems. Ragazzo [23] investigated the stability of sets that were generalizations of the simple pendulum double homoclinic loop. In our recent work [33, 34], codimension 2 bifurcations of double homoclinic loops and codimension 3 bifurcations of nontwisted double homoclinic loops with resonant eigenvalues were studied.

Bifurcations on inclination flips have been developed in homoclinic or heteroclinic loops. Homburg et al. [11] studied three parameter unfolding of resonant homoclinic orbits with orbit flip or inclination flip. Oldeman et al. [22] presented a numerical investigation of the unfolding for a specific three-dimensional vector field, which was

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constructed by Sandstede [24] to explicitly obtain the homoclinic loop with inclination flip and orbit flip. Shui et al. [26] studied codimension 3 nonresonant homoclinic orbit bifurcation with two inclination flips. However, there is no attention to the problem of double homoclinic loops with inclination flips. Motivated by this fact, we will study the problems of homoclinic and periodic orbits bifurcated from double homoclinic loops with W^s inclination flip of Γ_1 and nonresonant eigenvalues in four dimensional systems. Generally speaking, the bifurcation is more complicated as Γ is inclination flip and double homoclinic loops have higher codimension than a single homoclinic loop under the same conditions. Therefore, our work will be more difficult and challenging.

The rest of this paper is organized as follows. In Section 2, some hypotheses are given for our discussion and the normal form is established. In Section 3, the Poincaré map is set up and the bifurcation equations are given. In Section 4, by analysing bifurcation equations, the rich results of inclination flip bifurcations are obtained under different conditions. In Section 5, we give an analytical example to clarify our main results. A brief conclusion ends the paper in Section 6.

2. Hypotheses and Normal form

Consider the following C^r system and its unperturbed system

$$\dot{z} = f(z) + g(z, \nu), \quad (2.1)$$

$$\dot{z} = f(z), \quad (2.2)$$

where r is large enough, $z \in R^4$, $\nu \in R^l$, $l \geq 3$, $0 < |\nu| \ll 1$, $f(0) = 0$, $g(0, \nu) = g(z, 0) = 0$.

We make the following assumptions, which are shown in Figure 1.

(H_1) The linearization $Df(0)$ has simple real eigenvalues at the equilibrium 0: $-\rho_2, -\rho_1, \lambda_1, \lambda_2$ satisfying

$$-\rho_2 < -\rho_1 < 0 < \lambda_1 < \lambda_2 \quad \text{and} \quad \rho_1 > \lambda_1.$$

(H_2) System (1.2) has double homoclinic loops $\Gamma = \Gamma_1 \cup \Gamma_2, \Gamma_i = \{z = r_i(t) : t \in R, r_i(\pm\infty) = 0\}$ and $\dim(T_{r_i(t)}W^s \cap T_{r_i(t)}W^u) = 1$, $i = 1, 2$, where W^s and W^u are the stable and unstable manifolds of 0, respectively.

(H_3) Let $e_i^\pm = \lim_{t \rightarrow \mp\infty} \frac{\dot{r}_i(t)}{|\dot{r}_i(t)|}$, and $e_i^+ \in T_0W^u, e_i^- \in T_0W^s$ be unit eigenvectors corresponding to λ_1 and $-\rho_1$, respectively, and satisfying $e_1^+ = -e_2^+, e_1^- = -e_2^-$.

(H_4) $\text{Span}\{T_{r_i(t)}W^u, T_{r_i(t)}W^s, e_i^+\} = R^4$ as $t \gg 1$,
 $\text{Span}\{T_{r_1(t)}W^u, T_{r_1(t)}W^s, e_1^-\} = R^4$ as $t \ll -1$,
 $\text{Span}\{T_{r_2(t)}W^u, T_{r_2(t)}W^s, T_{r_1(t)}W^{ss}\} = R^4$ as $t \ll -1$.

The hypotheses (H_4) implies that W^s of Γ_1 is inclination flip. Furthermore, both W^s and W^u of Γ_2 as well as W^u of Γ_1 have the strong inclination property. That is to say, a general vector in $T_{r_2(t)}W^s$ (resp. $T_{r_i(t)}W^u$) not belonging to $\text{span}\{\dot{r}(t)\}$ should go to the strong stable (resp. unstable) direction as $t \rightarrow +\infty$ (resp. $t \rightarrow -\infty$).

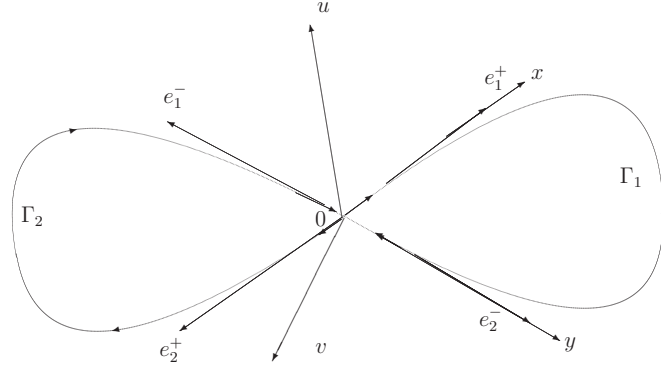


Figure 1. Double homoclinic loops $\Gamma = \Gamma_1 \cup \Gamma_2$

As shown in Figure 1, under the hypotheses $(H_1) - (H_4)$, we can see that the double homoclinic loops Γ are codimension 3.

The single homoclinic loop in high-dimensional systems has been investigated by many authors (see [1, 2, 4–6, 9, 11, 17, 22, 24, 26, 32, 35] and the references therein). In this paper, we only focus on bifurcations of the large loop, that is, the double loops $\Gamma = \Gamma_1 \cup \Gamma_2$.

We assume that $r \geq 3Q$ and $D^N f(0) = 0$ for $N = 0, 1$. It satisfies the Sternberg condition of order Q and K is the Q -smoothness of $Df(0) = 0$, where $Q = K([\frac{\lambda_2}{\lambda_1}] + [\frac{\rho_2}{\rho_1}] + 2)$, so system (2.1) is uniformly C^K linearizable according to [12]. Therefore, there exists U , a small neighborhood of 0 in R^4 , such that, for $\nu \in R^l$, $0 < |\nu| \ll 1$ and $\forall(x, y, u, v) \in U$, system (2.1) has the following C^{K-1} ($K \geq 4$) normal form

$$\begin{aligned} \dot{x} &= \lambda_1(\nu)x, \\ \dot{y} &= -\rho_1(\nu)y, \\ \dot{u} &= \lambda_2(\nu)u, \\ \dot{v} &= -\rho_2(\nu)v. \end{aligned} \tag{2.3}$$

3. Poincaré map and bifurcation equations

Now we consider the linear variational system of (2.2) and its adjoint system

$$\dot{z} = Df(r_i(t))z, \tag{3.1}$$

$$\dot{z} = -(Df(r_i(t)))^* z. \tag{3.2}$$

Denote $r_i(t) = (r_i^x(t), r_i^y(t), r_i^u(t), r_i^v(t))$. Under the coordinates corresponding to system (2.3), one can take T_i^0 and T_i^1 large enough such that $r_i(-T_i^1) = \{(-1)^{(i-1)}\delta, 0, 0, 0\}$, $r_i(T_i^0) = \{0, (-1)^{(i-1)}\delta, 0, 0\}$, where δ is small enough such that $\{(x, y, u, v) : |x|, |y|, |u|, |v| < 2\delta\} \subset U$.

Lemma 3.1. System (3.1) has a fundamental solution matrix

$$Z_i(t) = (z_i^1(t), z_i^2(t), z_i^3(t), z_i^4(t))$$

satisfying

$$\begin{aligned} z_i^1(t) &\in (T_{r_i(t)}W^u)^c \cap (T_{r_i(t)}W^s)^c, \\ z_i^2(t) &= (-1)^i r_i(t)/|r_i^y(T_i^0)| \in T_{r_i(t)}W^u \cap T_{r_i(t)}W^s, \\ z_i^3(t) &\in T_{r_i(t)}W^{uu}, \\ z_i^4(t) &\in T_{r_i(t)}W^s, \end{aligned}$$

$$\begin{aligned} Z_1(-T_1^1) &= \begin{pmatrix} w_1^{11} & w_1^{21} & 0 & w_1^{41} \\ 0 & 0 & 0 & w_1^{42} \\ w_1^{13} & 0 & 1 & w_1^{43} \\ \bar{w}_1^{14} & 0 & 0 & w_1^{44} \end{pmatrix}, & Z_1(T_1^0) &= \begin{pmatrix} 1 & 0 & w_1^{31} & 0 \\ 0 & 1 & w_1^{32} & 0 \\ 0 & 0 & w_1^{33} & 0 \\ w_1^{14} & 0 & w_1^{34} & 1 \end{pmatrix}, \\ Z_2(-T_2^1) &= \begin{pmatrix} w_2^{11} & w_2^{21} & 0 & w_2^{41} \\ w_2^{12} & 0 & 0 & w_2^{42} \\ w_2^{13} & 0 & 1 & w_2^{43} \\ 0 & 0 & 0 & w_2^{44} \end{pmatrix}, & Z_2(T_2^0) &= \begin{pmatrix} 1 & 0 & w_2^{31} & 0 \\ 0 & 1 & w_2^{32} & 0 \\ 0 & 0 & w_2^{33} & 0 \\ w_2^{14} & 0 & w_2^{34} & 1 \end{pmatrix}, \end{aligned}$$

where $\bar{w}_1^{14}w_2^{12}w_1^{42}w_2^{44}w_i^{33} \neq 0$, $w_i^{21} < 0$, $|w_i^{14}| \ll 1$, $|(\bar{w}_1^{14})^{-1}w_1^{1j}| \ll 1$, $j = 1, 3$, $|(w_2^{12})^{-1}w_2^{1j}| \ll 1$, $j = 1, 3$, $|(w_1^{42})^{-1}w_1^{4j}| \ll 1$, $j \neq 4$, $|(w_2^{44})^{-1}w_2^{4j}| \ll 1$, $j \neq 4$, $|(w_i^{33})^{-1}(w_i^{3j})| \ll 1$, $j \neq 3$, as $T_i^j \gg 1$ for $i = 1, 2, j = 0, 1$.

Proof. Due to the definition of $z_1^2(t) = -r_1(t)/|r_1^y(T_1^0)|$, we obtain the expression of $z_1^2(-T_1^1)$, $z_1^2(T_1^0)$ and $w_1^{21} < 0$. Owing to $\frac{\dot{r}_1(t)}{|r_1(t)|} \rightarrow e_1^+ \in T_0W^u$ (as $t \rightarrow -\infty$) and the hypotheses (H_4) , we know that $z_1^4(t)$ with $z_1^4(T_1^0) = (0, 0, 0, 1)^*$ approaches T_0W^{ss} asymptotically. Because W^s of Γ_1 is inclination flip, and therefore, $w_1^{42} \neq 0$. Similarly, we have $w_1^{33} \neq 0$. Take $\bar{z}_1^1(t) \in (T_{r_1(t)}W^u)^c \cap (T_{r_1(t)}W^s)^c$ such that $\bar{z}_1^1(T_1^0) = (1, 0, 0, 0)^*$, $\bar{z}_1^1(-T_1^1) = (w_0^{11}, w_0^{12}, w_0^{13}, w_0^{14})^*$. If $w_0^{12} = 0$, then we set $z_1^1(t) = \bar{z}_1^1(t)$. Otherwise, due to $w_1^{42} \neq 0$, we denote $z_1^1(t) \equiv \bar{z}_1^1(t) - w_0^{12}(w_1^{42})^{-1}z_1^4(t)$, then $z_1^1(t)$ satisfies the desired conditions at moments T_1^0 and $-T_1^1$. Based on $\det Z_1(-T_1^1) \neq 0$, $\bar{w}_1^{14} \neq 0$ is clear.

By the expressions of the local invariant manifolds in U , the values of $z_2^2(t)$, $z_2^3(t)$, $z_2^4(t)$ at $t = -T_2^1, T_2^0$ and $w_2^{21} < 0$ are clear. Owing to $\frac{\dot{r}_2(t)}{|r_2(t)|} \rightarrow e_2^- \in T_0W^s$ (as $t \rightarrow +\infty$) and the hypotheses (H_4) , W^s of Γ_2 has the strong inclination property. We know that $z_2^4(t)$ with $z_2^4(T_2^0) = (0, 0, 0, 1)^*$ approaches T_0W^{ss} asymptotically (as $t \rightarrow +\infty$), and therefore, $w_2^{44} \neq 0$. Similarly, we have $w_2^{33} \neq 0$. Take $\bar{z}_2^1(t) \in (T_{r_2(t)}W^u)^c \cap (T_{r_2(t)}W^s)^c$ such that $\bar{z}_2^1(T_2^0) = (1, 0, 0, 0)^*$, $\bar{z}_2^1(-T_2^1) = (\bar{w}_0^{11}, \bar{w}_0^{12}, \bar{w}_0^{13}, \bar{w}_0^{14})^*$. If $\bar{w}_0^{14} = 0$, then we set $z_2^1(t) = \bar{z}_2^1(t)$. Otherwise, due to $w_2^{44} \neq 0$, we denote $w_2^{14} = -(w_2^{44})^{-1}\bar{w}_0^{14}$, and $z_2^1(t) \equiv \bar{z}_2^1(t) + w_2^{14}z_2^4(t)$. Then $z_2^1(t)$ satisfies the desired conditions at moments T_2^0 and $-T_2^1$. Note that $w_2^{12} = 0$ means $z_2^1(t) \in T_{r_2(t)}W_2^u$, whereas, by definition, $z_2^1(t)$ should belong to $(T_{r_2(t)}W_2^u)^c$. So we have $w_2^{12} \neq 0$. The remainder is easy to check, we omit the detail. \square

Let $\Psi_i(t) = (Z_i^{-1}(t))^* = (\psi_i^1(t), \psi_i^2(t), \psi_i^3(t), \psi_i^4(t))$. Obviously, $\Psi_i(t)$ is a funda-

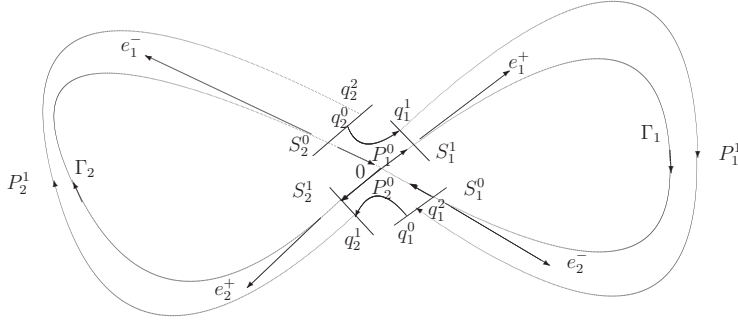


Figure 2. The Poincaré map

mental solution matrix of the adjoint system (3.2). Introduce the transformation

$$z = r_i(t) + (z_i^1(t), z_i^2(t), z_i^3(t), z_i^4(t)) \cdot (n_i^1, 0, n_i^3, n_i^4) \stackrel{def}{=} S_i(t) \quad (3.3)$$

in the neighborhood of Γ_i , and choose the cross sections (see Figure 2)

$$S_i^0 = \{z = S_i(T_i^0) : |x|, |y|, |u|, |v| < 2\delta\} \subset U,$$

$$S_i^1 = \{z = S_i(-T_i^1) : |x|, |y|, |u|, |v| < 2\delta\} \subset U$$

for $i = 1, 2$, then system (2.1) takes the following normal form

$$\dot{n}_i^j = (\psi_i^j(t))^* g_\nu(r_i(t), 0)\nu + h.o.t., \quad -T_i^1 \leq t \leq T_i^0, \quad i = 1, 2; j = 1, 3, 4. \quad (3.4)$$

By integrating both sides from $-T_i^1$ to T_i^0 , equation (3.4) produces a map $P_i^1 : S_i^1 \rightarrow S_i^0$ as follows.

$$n_i^j(T_i^0) = n_i^j(-T_i^1) + M_i^j \nu + h.o.t., \quad i = 1, 2; j = 1, 3, 4, \quad (3.5)$$

where $M_i^j = \int_{-T_i^1}^{T_i^0} (\psi_i^j(t))^* g_\nu(r_i(t), 0) dt$, $i = 1, 2; j = 1, 3, 4$ are Melnikov vectors.

Lemma 3.2. $M_i^1 = \int_{-T_i^1}^{T_i^0} (\psi_i^1(t))^* g_\nu(r_i(t), 0) dt = \int_{-\infty}^{+\infty} (\psi_i^1(t))^* g_\nu(r_i(t), 0) dt$, $i = 1, 2$.

Proof. Without loss of generality, it is sufficient to verify that $(\psi_1^1(t))^* g_\nu(r_1(t), 0) = 0$ for $t \geq T_1^0$ and $t \leq -T_1^1$. We have $r_1(t) = (0, r_1^y(t), 0, 0)$ with $|r_1^y(t)| = O(\delta e^{-\rho_1(t-T_1^0)}) < \delta$ for $t \geq T_1^0$ and $r_1(t) = (r_1^x(t), 0, 0, 0)$ with $|r_1^x(t)| = O(\delta e^{\lambda_1(t+T_1^1)}) < \delta$ for $t \leq -T_1^1$. According to the normal form (2.3), we have

$$g_\nu(r_1(t), 0) = (0, O(\delta), 0, 0), \quad \text{for } t \geq T_1^0, \quad g_\nu(r_1(t), 0) = (O(\delta), 0, 0, 0), \quad \text{for } t \leq -T_1^1.$$

Since $(\Psi_1(t))^* Z_1(t) = I$, we have $(\psi_1^j(t))^* z_1^1(t) = 0$, $j=2, 3, 4$. Denote

$$(\psi_1^j(t))^* = (\psi_1^{j1}(t), \psi_1^{j2}(t), \psi_1^{j3}(t), \psi_1^{j4}(t)),$$

then $z_1^1(T_1^0) = (1, 0, 0, w_1^{14})^*$ implies that $\psi_1^{j1}(T_1^0) = 0$, $j = 2, 3, 4$. Thereafter, we have $\psi_1^{j1}(t) = 0$, for $t > T_1^0$, $j = 2, 3, 4$, since $Df(r_1(t))$ and its adjoint matrix are both diagonal. Likewise, we can also obtain $\psi_1^{j4}(-T_1^1) = 0$, $j = 2, 3, 4$. Consequently, $\psi_1^{j4}(t) = 0$, for $t < -T_1^1$, $j = 2, 3, 4$. Thus, we have $(\psi_1^1(t))^* g_\nu(r_1(t), 0) = 0$ for $t \geq T_1^0$ and $t \leq -T_1^1$. \square

Next we consider the maps $P_1^0 : S_2^0 \rightarrow S_1^1$; $q_2^0 \mapsto q_1^1$ and $P_2^0 : S_1^0 \rightarrow S_2^1$; $q_1^0 \mapsto q_2^1$ induced by the flow of (2.3) in the neighborhood U of $z = 0$. Set the flying time from q_2^0 to q_1^1 as τ_1 , q_1^0 to q_2^1 as τ_2 , and the Silnikov time $s_k = e^{-\lambda_1(\nu)\tau_k}$, $k = 1, 2$. Then we have

$$\begin{aligned} P_1^0 : q_2^0(x_2^0, y_2^0, u_2^0, v_2^0) &\mapsto q_1^1(x_1^1, y_1^1, u_1^1, v_1^1) \\ x_2^0 &= s_1 x_1^1, \\ y_1^1 &= s_1^{\rho_1/\lambda_1} y_2^0, \\ u_2^0 &= s_1^{\lambda_2/\lambda_1} u_1^1, \\ v_1^1 &= s_1^{\rho_2/\lambda_1} v_2^0, \end{aligned} \quad (3.6)$$

$$\begin{aligned} P_2^0 : q_1^0(x_1^0, y_1^0, u_1^0, v_1^0) &\mapsto q_2^1(x_2^1, y_2^1, u_2^1, v_2^1) \\ x_1^0 &= s_2 x_2^1, \\ y_2^1 &= s_2^{\rho_1/\lambda_1} y_1^0, \\ u_1^0 &= s_2^{\lambda_2/\lambda_1} u_2^1, \\ v_2^1 &= s_2^{\rho_2/\lambda_1} v_1^0. \end{aligned} \quad (3.7)$$

Then, we seek the relations between $q_i^{2j}(x_i^{2j}, y_i^{2j}, u_i^{2j}, v_i^{2j}) \in S_i^0$, $q_i^{2j+1}(x_i^{2j+1}, y_i^{2j+1}, u_i^{2j+1}, v_i^{2j+1}) \in S_i^1$, $P_i^0(q_i^{2j}) = q_i^{2j+1}$ and their new coordinates $N_i^{2j} = (n_i^{2j,1}, 0, n_i^{2j,3}, n_i^{2j,4})$, $N_i^{2j+1} = (n_i^{2j+1,1}, 0, n_i^{2j+1,3}, n_i^{2j+1,4})$ for $i = 1, 2$, where $q_3^0 = q_1^0$. Using (3.3) and the expression of $Z_i(-T_i^1)$ and $Z_i(T_i^0)$, we achieve

$$\begin{aligned} n_1^{2j,1} &= x_1^{2j} - w_1^{31}(w_1^{33})^{-1} u_1^{2j}, \\ n_1^{2j,3} &= (w_1^{33})^{-1} u_1^{2j}, \\ n_1^{2j,4} &= v_1^{2j} - w_1^{14} x_1^{2j} + (w_1^{14} w_1^{31} - w_1^{34})(w_1^{33})^{-1} u_1^{2j}, \end{aligned} \quad (3.8)$$

$$\begin{aligned} n_1^{2j+1,1} &= (\bar{w}_1^{14})^{-1} v_1^{2j+1} - (\bar{w}_1^{14})^{-1} w_1^{44} (w_1^{42})^{-1} v_1^{2j+1}, \\ n_1^{2j+1,3} &= u_1^{2j+1} - w_1^{43} (w_1^{42})^{-1} y_1^{2j+1} + w_1^{13} (w_1^{14})^{-1} (w_1^{42})^{-1} w_1^{44} y_1^{2j+1} - w_1^{13} (\bar{w}_1^{14})^{-1} v_1^{2j+1}, \\ n_1^{2j+1,4} &= (w_1^{42})^{-1} y_1^{2j+1}, \end{aligned} \quad (3.9)$$

$$\begin{aligned} n_2^{2j,1} &= x_2^{2j} - w_2^{31}(w_2^{33})^{-1} u_2^{2j}, \\ n_2^{2j,3} &= (w_2^{33})^{-1} u_2^{2j}, \\ n_2^{2j,4} &= v_2^{2j} - w_2^{14} x_2^{2j} + (w_2^{14} w_2^{31} - w_2^{34})(w_2^{33})^{-1} u_2^{2j}, \end{aligned} \quad (3.10)$$

$$\begin{aligned} n_2^{2j+1,1} &= (w_2^{12})^{-1} y_2^{2j+1} - (w_2^{12})^{-1} w_2^{42} (w_2^{44})^{-1} v_2^{2j+1}, \\ n_2^{2j+1,3} &= u_2^{2j+1} - w_2^{13} (w_2^{12})^{-1} y_2^{2j+1} + (w_2^{13} (w_2^{12})^{-1} w_2^{42} - w_2^{43})(w_2^{44})^{-1} v_2^{2j+1}, \\ n_2^{2j+1,4} &= (w_2^{44})^{-1} v_2^{2j+1}, \end{aligned} \quad (3.11)$$

and

$$x_i^{2j+1} \approx (-1)^{i-1} \delta, \quad y_i^{2j} \approx (-1)^{i-1} \delta. \quad (3.12)$$

Notice that $n_i^j(T_i^0) = n_i^{2,j}$, $n_i^j(-T_i^1) = n_i^{1,j}$, now we are ready to give the Poincaré maps from (3.5)-(3.12):

$$\begin{aligned} F_1 &= P_1^1 \circ P_1^0 : S_2^0 \rightarrow S_1^0, \\ n_1^{2,1} &= (\bar{w}_1^{14})^{-1} s_1^{\rho_2/\lambda_1} v_2^0 + (\bar{w}_i^{14})^{-1} w_i^{44} (w_i^{42})^{-1} \delta s_1^{\rho_1/\lambda_1} + M_1^1 \nu + h.o.t., \\ n_1^{2,3} &= u_1^1 + w_1^{43} (w_1^{42})^{-1} \delta s_1^{\rho_1/\lambda_1} + w_1^{13} (\bar{w}_1^{14})^{-1} (w_1^{42})^{-1} w_1^{44} \delta s_1^{\rho_1/\lambda_1} \\ &\quad - w_1^{13} (\bar{w}_1^{14})^{-1} s_1^{\rho_2/\lambda_1} v_2^0 + M_1^3 \nu + h.o.t., \\ n_1^{2,4} &= (w_1^{42})^{-1} \delta s_1^{\rho_1/\lambda_1} + M_1^4 \nu + h.o.t., \end{aligned} \quad (3.13)$$

$$\begin{aligned} F_2 &= P_2^1 \circ P_2^0 : S_1^0 \rightarrow S_2^0, \\ n_2^{2,1} &= (w_2^{12})^{-1} \delta s_2^{\rho_1/\lambda_1} - (w_2^{12})^{-1} w_2^{42} (w_2^{44})^{-1} s_2^{\rho_2/\lambda_1} v_1^0 + M_2^1 \nu + h.o.t., \\ n_2^{2,3} &= u_2^1 - w_2^{13} (w_2^{12})^{-1} \delta s_2^{\rho_1/\lambda_1} + [w_2^{13} (w_2^{12})^{-1} w_2^{42} - w_2^{43}] (w_2^{44})^{-1} s_2^{\rho_2/\lambda_1} v_1^0 \\ &\quad + M_2^3 \nu + h.o.t., \\ n_2^{2,4} &= (w_2^{44})^{-1} s_2^{\rho_2/\lambda_1} v_1^0 + M_2^4 \nu + h.o.t.. \end{aligned} \quad (3.14)$$

Finally, equalities (3.6)-(3.11), (3.13) and (3.14) yield the successor function

$$G(s_1, s_2, u_1^1, u_2^1, v_1^0, v_2^0) = (G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4) = (F_1(q_2^0) - q_1^0, F_2(q_1^0) - q_2^0)$$

as follows.

$$\begin{aligned} G_1^1 &= (\bar{w}_1^{14})^{-1} s_1^{\rho_2/\lambda_1} v_2^0 + (\bar{w}_1^{14})^{-1} w_1^{44} (w_1^{42})^{-1} \delta s_1^{\rho_1/\lambda_1} + \delta s_2 \\ &\quad + w_1^{31} (w_1^{33})^{-1} s_2^{\lambda_2/\lambda_1} u_2^1 + M_1^1 \nu + h.o.t., \\ G_1^3 &= u_1^1 - w_1^{13} (\bar{w}_1^{14})^{-1} s_1^{\rho_2/\lambda_1} v_2^0 - [w_1^{13} (\bar{w}_1^{14})^{-1} w_1^{44} - w_1^{43}] (w_1^{42})^{-1} \delta s_1^{\rho_1/\lambda_1} \\ &\quad - (w_1^{33})^{-1} s_2^{\lambda_2/\lambda_1} u_2^1 + M_1^3 \nu + h.o.t., \\ G_1^4 &= -v_1^0 - (w_1^{42})^{-1} \delta s_1^{\rho_1/\lambda_1} - w_1^{14} \delta s_2 \\ &\quad - [w_1^{14} w_1^{31} - w_1^{34}] (w_1^{33})^{-1} s_2^{\lambda_2/\lambda_1} u_2^1 + M_1^4 \nu + h.o.t., \\ G_2^1 &= (w_2^{12})^{-1} \delta s_2^{\rho_1/\lambda_1} - (w_2^{12})^{-1} w_2^{42} (w_2^{44})^{-1} s_2^{\rho_2/\lambda_1} v_1^0 - \delta s_1 \\ &\quad + w_2^{31} (w_2^{33})^{-1} s_1^{\lambda_2/\lambda_1} u_1^1 + M_2^1 \nu + h.o.t., \\ G_2^3 &= u_2^1 - w_2^{13} (w_2^{12})^{-1} \delta s_2^{\rho_1/\lambda_1} + [w_2^{13} (w_2^{12})^{-1} w_2^{42} - w_2^{43}] (w_2^{44})^{-1} s_2^{\rho_2/\lambda_1} v_1^0 \\ &\quad - (w_2^{33})^{-1} s_1^{\lambda_2/\lambda_1} u_1^1 + M_2^3 \nu + h.o.t., \\ G_2^4 &= -v_2^0 + (w_2^{44})^{-1} s_2^{\rho_2/\lambda_1} v_1^0 + w_2^{14} \delta s_1 \\ &\quad - [w_2^{14} w_2^{31} - w_2^{34}] (w_2^{33})^{-1} s_1^{\lambda_2/\lambda_1} u_1^1 + M_2^4 \nu + h.o.t.. \end{aligned}$$

Clearly, there is an 1 – 1 correspondence between the large 1-periodic, large 1-homoclinic and large loop consisting of double homoclinic orbits of system (2.1) and the solution $Q = (s_1, s_2, u_1^1, u_2^1, v_1^0, v_2^0)$ of

$$(G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4) = 0 \quad (3.15)$$

with $s_1 \geq 0, s_2 \geq 0$.

For $G_1^4 = 0$ we get

$$v_1^0 = -(w_1^{42})^{-1} \delta s_1^{\rho_1/\lambda_1} - w_1^{14} \delta s_2 + M_1^4 \nu + O(s_2), \quad (3.16)$$

substituting v_1^0 into $G_2^4 = 0$ we have

$$v_2^0 = (w_2^{44})^{-1} M_1^4 \nu s_2^{\rho_2/\lambda_1} + w_2^{14} \delta s_1 + M_2^4 \nu + O(s_1^{\rho_1/\lambda_1}) + O(s_2^{\rho_2/\lambda_1+1}),$$

by substituting it into $(G_1^1, G_2^1) = 0$, we obtain the bifurcation equations

$$\begin{aligned} s_2 &= -(\bar{w}_1^{14})^{-1} M_2^4 \nu \delta^{-1} s_1^{\rho_2/\lambda_1} - (\bar{w}_1^{14})^{-1} w_2^{14} s_1^{\rho_2/\lambda_1+1} - \delta^{-1} M_1^1 \nu + h.o.t., \\ s_1 &= (w_2^{12})^{-1} s_2^{\rho_1/\lambda_1} + \delta^{-1} M_2^1 \nu + h.o.t.. \end{aligned} \quad (3.17)$$

4. Main results and their proofs

In this section, we study the existence, uniqueness and incoexistence of the double homoclinic loops, large 1-homoclinic orbit and large 1-periodic orbit.

Theorem 4.1. Suppose that $(H_1) - (H_4)$ hold. Then, for $|\nu|$ small enough, system (2.1) has at most one large loop consisting of two homoclinic orbits, one large 1-homoclinic orbit or one large 1-periodic orbit in a small neighborhood of Γ , and these orbits can not coexist.

Proof. We have

$$W = \frac{\partial(G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4)}{\partial Q} \Big|_{Q=0, \nu=0} = \begin{pmatrix} 0 & 0 & 0 & \delta & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -w_1^{14} \delta & 0 & 0 \\ -\delta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ w_2^{14} \delta & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

where $Q = Q(s_1, u_1^1, v_1^0, s_2, u_2^1, v_2^0)$ and $\det W = -\delta^2 \neq 0$. So the implicit function theorem says that, in the neighborhood of $(Q, \nu) = (0, 0)$, there exists a unique group of functions

$$s_i = s_i(\nu), \quad u_i^1 = u_i^1(\nu), \quad v_i^0 = v_i^0(\nu), \quad i = 1, 2,$$

satisfying $s_i(\nu) = u_i^1(\nu) = v_i^0(\nu) = 0$ as $\nu = 0$. Then if $s_1(\nu) = s_2(\nu) = 0$, system (2.1) has a unique large loop, that is, the loop Γ is persistent. If $s_1(\nu) > 0, s_2(\nu) = 0$ (or $s_1(\nu) = 0, s_2(\nu) > 0$), system (2.1) has a unique large 1-homoclinic orbit. If $s_1(\nu) > 0, s_2(\nu) > 0$, system (2.1) has a unique large 1-periodic orbit. \square

Theorem 4.2. Suppose that $(H_1) - (H_4)$ hold, then the following conclusions are true.

- (1) If $M_i^1 \neq 0$, then there exists a unique surface $\Sigma_i \stackrel{def}{=} \{\nu : M_i^1 \nu + h.o.t. = 0\}$ with codimension 1 and normal vectors M_i^1 at $\nu = 0$, such that system (2.1) has a homoclinic loop near Γ_i if and only if $\nu \in \Sigma_i$ and $|\nu| \ll 1$, that is, Γ_i is persistent.

- (2) If $\text{rank}(M_1^1, M_2^1) = 2$, then $\Sigma_{12} = \Sigma_1 \cap \Sigma_2$ is a codimension 2 surface and $0 \in \Sigma_{12}$ such that system (2.1) has a large loop consisting of two homoclinic orbits near Γ as $\nu \in \Sigma_{12}$ and $|\nu| \ll 1$, that is, Γ is persistent.

Proof. If (3.17) has solution $s_1 = s_2 = 0$, then we have

$$M_i^1 \nu + h.o.t. = 0, \quad i = 1, 2. \quad (4.1)$$

If $M_i^1 \neq 0$, then there exists a codimension 1 surface Σ_i with normal vector M_i^1 at $\nu = 0$, such that (3.17) has solution $s_1 = s_2 = 0$ as $\nu \in \Sigma_i$ and $|\nu| \ll 1$.

If $\text{rank}(M_1^1, M_2^1) = 2$, then $\Sigma_{12} = \Sigma_1 \cap \Sigma_2$ is a codimension 2 surface with normal plane $\text{span}\{M_1^1, M_2^1\}$ such that (3.17) has solution $s_1 = s_2 = 0$ as $\nu \in \Sigma_{12}$ and $|\nu| \ll 1$, equivalently, the large loop $\Gamma = \Gamma_1 \cup \Gamma_2$ is persistent. \square

Next, we discuss if (3.17) has solution $s_1 \geq 0, s_2 \geq 0$, then we have the following theorem.

Theorem 4.3. Suppose that $(H_1) - (H_4)$ hold, $|\nu| \ll 1$, and $\text{rank}(M_1^1, M_2^1) = 2$, then the following results are true.

- (1) In case $M_2^1 \nu < 0$ and $w_2^{12} < 0$, system (2.1) has no large 1-periodic orbit near Γ .
- (2) In case $M_2^1 \nu > 0$ and $w_2^{12} > 0$, system (2.1) has a unique large 1-periodic orbit near Γ as $\delta^{-1} M_1^1 \nu < g(\nu)$, a unique large 1-homoclinic orbit near Γ as $\delta^{-1} M_1^1 \nu = g(\nu)$, and no large 1-periodic orbit as $\delta^{-1} M_1^1 \nu > g(\nu)$, where $g(\nu) = -(\bar{w}_1^{14})^{-1} M_2^4 \nu \delta^{-1} (\delta^{-1} M_2^1 \nu)^{\rho_2 / \lambda_1} - (\bar{w}_1^{14})^{-1} w_2^{14} (\delta^{-1} M_2^1 \nu)^{\rho_2 / \lambda_1 + 1} + h.o.t.$.
- (3) In case $M_2^1 \nu < 0$ and $w_2^{12} > 0$, system (2.1) has a unique large 1-periodic orbit near Γ as $\delta^{-1} M_2^1 \nu > -(w_2^{12})^{-1} (-\delta^{-1} M_1^1 \nu)^{\rho_1 / \lambda_1} + h.o.t.$, a unique large 1-homoclinic orbit near Γ as $\delta^{-1} M_2^1 \nu = -(w_2^{12})^{-1} (-\delta^{-1} M_1^1 \nu)^{\rho_1 / \lambda_1} + h.o.t.$, and no large 1-periodic orbit as $\delta^{-1} M_2^1 \nu < -(w_2^{12})^{-1} (-\delta^{-1} M_1^1 \nu)^{\rho_1 / \lambda_1} + h.o.t.$.
- (4) In case $M_2^1 \nu > 0$ and $w_2^{12} < 0$, we have
 - (I) If $M_1^1 \nu > 0, (\bar{w}_1^{14})^{-1} M_2^4 \nu > 0$ and $(\bar{w}_1^{14})^{-1} w_2^{14} > 0$, then system (2.1) has no large 1-periodic orbit.
 - (II) If $M_1^1 \nu < 0, (\bar{w}_1^{14})^{-1} M_2^4 \nu < 0$ and $(\bar{w}_1^{14})^{-1} w_2^{14} < 0$, then system (2.1) has a unique large 1-periodic orbit as $-(w_2^{12})^{-1} (-\delta^{-1} M_1^1 \nu)^{\rho_1 / \lambda_1} + h.o.t. > \delta^{-1} M_2^1 \nu$, a unique large 1-homoclinic orbit as $-(w_2^{12})^{-1} (-\delta^{-1} M_1^1 \nu)^{\rho_1 / \lambda_1} + h.o.t. = \delta^{-1} M_2^1 \nu$, and no large 1-periodic orbit as $-(w_2^{12})^{-1} (-\delta^{-1} M_1^1 \nu)^{\rho_1 / \lambda_1} + h.o.t. < \delta^{-1} M_2^1 \nu$.
 - (III) If $M_1^1 \nu < 0, (\bar{w}_1^{14})^{-1} M_2^4 \nu > 0, (\bar{w}_1^{14})^{-1} w_2^{14} > 0$ and $|M_1^1 \nu|^{\frac{\alpha-1}{\alpha}} \ll |M_2^4 \nu|$ (as $\nu \rightarrow 0$), then system (2.1) has a unique large 1-periodic orbit as $\delta^{-1} M_2^1 \nu > -(w_2^{12})^{-1} (-\delta^{-1} M_1^1 \nu)^{\rho_1 / \lambda_1} + h.o.t.$ or $\delta^{-1} M_2^1 \nu \leq (-\bar{w}_1^{14} M_1^1 \nu / M_2^4 \nu)^{\lambda_1 / \rho_2} + h.o.t.$, a unique large 1-homoclinic orbit as $\delta^{-1} M_2^1 \nu = -(w_2^{12})^{-1} (-\delta^{-1} M_1^1 \nu)^{\rho_1 / \lambda_1} + h.o.t.$, and no large 1-periodic orbit as $\delta^{-1} M_2^1 \nu < -(w_2^{12})^{-1} (-\delta^{-1} M_1^1 \nu)^{\rho_1 / \lambda_1} + h.o.t.$ or $\delta^{-1} M_2^1 \nu > (-\bar{w}_1^{14} M_1^1 \nu / M_2^4 \nu)^{\lambda_1 / \rho_2} + h.o.t.$.
 - (IV) If $M_1^1 \nu > 0, (\bar{w}_1^{14})^{-1} M_2^4 \nu < 0$ and $(\bar{w}_1^{14})^{-1} w_2^{14} < 0$, then system (2.1) has a unique large 1-periodic orbit as $\delta^{-1} M_1^1 \nu < h(\nu)$, a unique large 1-homoclinic orbit as $\delta^{-1} M_1^1 \nu = h(\nu)$, and no large 1-periodic orbit as $\delta^{-1} M_1^1 \nu > h(\nu)$, where $h(\nu) = -(\bar{w}_1^{14})^{-1} M_2^4 \nu \delta^{-1} (\delta^{-1} M_2^1 \nu)^{\rho_2 / \lambda_1} - (\bar{w}_1^{14})^{-1} w_2^{14} (\delta^{-1} M_2^1 \nu)^{\rho_2 / \lambda_1 + 1} + h.o.t.$.

- (V) If $w_2^{14}M_2^4\nu < 0$, then we can get the following results.
- (i) If $D(\nu) > 0$ and $F(\nu) < 0$, then system (2.1) has a unique large 1-periodic orbit as $s_{1k_+}(\nu) < \delta^{-1}M_2^1\nu + h.o.t.$, a unique large 1-homoclinic orbit as $s_{1k_+}(\nu) = \delta^{-1}M_2^1\nu + h.o.t.$, and no large 1-periodic orbit as $s_{1k_+}(\nu) > \delta^{-1}M_2^1\nu + h.o.t.$
 - (ii) If $D(\nu) > 0$ and $\Delta(\nu) < 0$, then system (2.1) has no large 1-periodic orbit as $F(\nu) > (-\delta^{-1}w_2^{12}M_2^1\nu)^{\lambda_1/\rho_1} + h.o.t.$, a unique large 1-homoclinic orbit as $F(\nu) = (-\delta^{-1}w_2^{12}M_2^1\nu)^{\lambda_1/\rho_1} + h.o.t.$, and a unique large 1-periodic orbit as $F(\nu) < (-\delta^{-1}w_2^{12}M_2^1\nu)^{\lambda_1/\rho_1} + h.o.t.$
 - (iii) If $D(\nu) > 0$, $F(\nu) > 0$ and $\Delta(\nu) > 0$, then system (2.1) has no large 1-periodic orbit as $F(\nu) > (-\delta^{-1}w_2^{12}M_2^1\nu)^{\lambda_1/\rho_1} + h.o.t.$, a unique large 1-homoclinic orbit as $F(\nu) = (-\delta^{-1}w_2^{12}M_2^1\nu)^{\lambda_1/\rho_1} + h.o.t.$, a unique large 1-periodic orbit as $0 < F(\nu) < (-\delta^{-1}w_2^{12}M_2^1\nu)^{\lambda_1/\rho_1} + h.o.t.$ and $s_{1k_-}(\nu) > \delta^{-1}M_2^1\nu + h.o.t.$ ($s_{1k_+}(\nu) < \delta^{-1}M_2^1\nu + h.o.t.$), no large 1-periodic orbit as $s_{1k_-}(\nu) < \delta^{-1}M_2^1\nu + h.o.t. < s_{1k_+}(\nu)$, and a large 1-homoclinic orbit as $s_{1k_-}(\nu) = \delta^{-1}M_2^1\nu + h.o.t.$ ($s_{1k_+}(\nu) = \delta^{-1}M_2^1\nu + h.o.t.$).
 - (iv) If $D(\nu) < 0$ and $\Delta(\nu) < 0$, then system (2.1) has no large 1-periodic orbit.
 - (v) If $D(\nu) < 0$, $F(\nu) < 0$ and $\Delta(\nu) > 0$, then system (2.1) has no large 1-periodic orbit as $s_{1k_-}(\nu) > \delta^{-1}M_2^1\nu + h.o.t.$ or $s_{1k_+}(\nu) < \delta^{-1}M_2^1\nu + h.o.t.$, a unique large 1-homoclinic orbit as $s_{1k_-}(\nu) = \delta^{-1}M_2^1\nu + h.o.t.$ or $s_{1k_+}(\nu) = \delta^{-1}M_2^1\nu + h.o.t.$, and a unique large 1-periodic orbit as $s_{1k_-}(\nu) < \delta^{-1}M_2^1\nu + h.o.t. < s_{1k_+}(\nu)$.
 - (vi) If $D(\nu) < 0$ and $F(\nu) > 0$, then system (2.1) has no large 1-periodic orbit as $F(\nu) > (-\delta^{-1}w_2^{12}M_2^1\nu)^{\lambda_1/\rho_1} + h.o.t.$, a unique large 1-homoclinic orbit as $F(\nu) = (-\delta^{-1}w_2^{12}M_2^1\nu)^{\lambda_1/\rho_1} + h.o.t.$, a unique large 1-periodic orbit as $0 < F(\nu) < (-\delta^{-1}w_2^{12}M_2^1\nu)^{\lambda_1/\rho_1} + h.o.t.$ and $s_{1k_+}(\nu) > \delta^{-1}M_2^1\nu + h.o.t.$, a unique large 1-homoclinic orbit as $s_{1k_+}(\nu) = \delta^{-1}M_2^1\nu + h.o.t.$, and no large 1-periodic orbit as $s_{1k_+}(\nu) < \delta^{-1}M_2^1\nu + h.o.t.$

Where

$$\begin{aligned}
D(\nu) &= \frac{\rho_2 M_2^4 \nu}{2\lambda_1 \delta \bar{w}_1^{14}} \left(-\frac{\rho_2 M_2^4 \nu}{(\rho_2 + \lambda_1) \delta w_2^{14}} \right)^{\rho_2/\lambda_1 - 2} + h.o.t., \\
E(\nu) &= -\frac{\rho_2 M_2^4 \nu}{\lambda_1 \delta \bar{w}_1^{14}} \left(-\frac{\rho_2 M_2^4 \nu}{(\rho_2 + \lambda_1) \delta w_2^{14}} \right)^{\rho_2/\lambda_1 - 1} + h.o.t., \\
F(\nu) &= -\delta^{-1} M_1^1 \nu - \frac{(2\lambda_1^2 - \rho_2^2 - \lambda_1 \rho_2) M_2^4 \nu}{2\lambda_1 (\rho_2 + \lambda_1) \delta \bar{w}_1^{14}} \left(-\frac{\rho_2 M_2^4 \nu}{(\rho_2 + \lambda_1) \delta w_2^{14}} \right)^{\rho_2/\lambda_1} + h.o.t., \\
s_{1k_{\pm}}(\nu) &= \frac{-E(\nu) \pm \sqrt{E^2(\nu) - 4D(\nu)F(\nu)}}{2D(\nu)} + h.o.t., \\
\Delta(\nu) &= E^2(\nu) - 4D(\nu)F(\nu).
\end{aligned}$$

Proof.

- (1) We can obtain it easily from the second equation of (3.17).

- (2) In case $M_2^1\nu > 0$, $w_2^{12} > 0$, we have $s_1 > 0$ as $s_2 \geq 0$. Eliminating s_1 , (3.17) is reduced to

$$\begin{aligned} F(s_2) &\triangleq s_2 + (\bar{w}_1^{14})^{-1}M_2^4\nu\delta^{-1}(\delta^{-1}M_2^1\nu + (w_2^{12})^{-1}s_2^{\rho_2/\lambda_1})^{\rho_2/\lambda_1} \\ &\quad + (\bar{w}_1^{14})^{-1}w_2^{14}(\delta^{-1}M_2^1\nu + (w_2^{12})^{-1}s_2^{\rho_2/\lambda_1})^{\rho_2/\lambda_1+1} + \delta^{-1}M_1^1\nu + h.o.t. \\ &= 0, \end{aligned}$$

For $(\rho_2/\lambda_1) > (\rho_1/\lambda_1) > 1$, we get $F'(s_2) \approx 1 > 0$,

$$\begin{aligned} F(0) &= (\bar{w}_1^{14})^{-1}M_2^4\nu\delta^{-1}(\delta^{-1}M_2^1\nu)^{\rho_2/\lambda_1} \\ &\quad + (\bar{w}_1^{14})^{-1}w_2^{14}(\delta^{-1}M_2^1\nu)^{\rho_2/\lambda_1+1} + \delta^{-1}M_1^1\nu + h.o.t. \\ &\triangleq \delta^{-1}M_1^1\nu - g(\nu). \end{aligned}$$

If $\delta^{-1}M_1^1\nu < g(\nu)$, then $F(s_2) = 0$ has a unique small positive solution. If $\delta^{-1}M_1^1\nu = g(\nu)$, then $F(s_2) = 0$ has a unique solution $s_2 = 0$. If $\delta^{-1}M_1^1\nu > g(\nu)$, then $F(s_2) = 0$ has no positive solution.

- (3) In case $M_2^1\nu < 0$, $w_2^{12} > 0$, we get $s_2 > 0$ as $s_1 \geq 0$. Eliminating s_2 , (3.17) is reduce to

$$\begin{aligned} F(s_1) &\triangleq s_1 - (w_2^{12})^{-1}(-(\bar{w}_1^{14})^{-1}M_2^4\nu\delta^{-1}s_1^{\rho_2/\lambda_1} - (\bar{w}_1^{14})^{-1}w_2^{14}s_1^{\rho_2/\lambda_1+1} \\ &\quad - \delta^{-1}M_1^1\nu)^{\rho_1/\lambda_1} - \delta^{-1}M_2^1\nu + h.o.t. \\ &= 0, \end{aligned}$$

For $(\rho_2/\lambda_1) > (\rho_1/\lambda_1) > 1$, we get $F'(s_1) \approx 1 > 0$,

$$F(0) = - (w_2^{12})^{-1}(-\delta^{-1}M_1^1\nu)^{\rho_1/\lambda_1} - \delta^{-1}M_2^1\nu + h.o.t..$$

If $\delta^{-1}M_2^1\nu > -(w_2^{12})^{-1}(-\delta^{-1}M_1^1\nu)^{\rho_1/\lambda_1} + h.o.t.$, then $F(s_1) = 0$ has a unique small positive solution. If $\delta^{-1}M_2^1\nu = -(w_2^{12})^{-1}(-\delta^{-1}M_1^1\nu)^{\rho_1/\lambda_1} + h.o.t.$, then $F(s_1) = 0$ has a unique solution $s_1 = 0$. If $\delta^{-1}M_2^1\nu < -(w_2^{12})^{-1}(-\delta^{-1}M_1^1\nu)^{\rho_1/\lambda_1} + h.o.t.$, then $F(s_1) = 0$ has no positive solution.

- (4) (I) For $M_1^1\nu > 0$, $(\bar{w}_1^{14})^{-1}M_2^4\nu > 0$ and $(\bar{w}_1^{14})^{-1}w_2^{14} > 0$, it is easy to get the result from the first equation of (3.17).
 (II) For $M_1^1\nu < 0$, $(\bar{w}_1^{14})^{-1}M_2^4\nu < 0$ and $(\bar{w}_1^{14})^{-1}w_2^{14} < 0$, the first equation of (3.17) shows that $s_2 > 0$ as $s_1 \geq 0$, so we can eliminate s_2 and reduce (3.17) to $L_1(s_1) = K_1(s_1)$, where

$$\begin{aligned} L_1(s_1) &= [w_2^{12}(s_1 - \delta^{-1}M_2^1\nu)]^{\lambda_1/\rho_1} + h.o.t., \\ K_1(s_1) &= -(\bar{w}_1^{14})^{-1}M_2^4\nu\delta^{-1}s_1^{\rho_2/\lambda_1} - (\bar{w}_1^{14})^{-1}w_2^{14}s_1^{\rho_2/\lambda_1+1} - \delta^{-1}M_1^1\nu + h.o.t., \\ L_1(0) &= [w_2^{12}(-\delta^{-1}M_2^1\nu)]^{\lambda_1/\rho_1} + h.o.t., \\ K_1(0) &= -\delta^{-1}M_1^1\nu + h.o.t.. \end{aligned}$$

Because of $L_1'(s_1) < 0 < K_1'(s_1)$, if $L_1(0) > K_1(0)$, we can see that $L_1(s_1) = K_1(s_1)$ has a unique small positive solution. If $L_1(0) = K_1(0)$, then $L_1(s_1) = K_1(s_1)$ has a unique solution $s_1 = 0$. If $L_1(0) < K_1(0)$, then $L_1(s_1) = K_1(s_1)$ has no positive solution.

- (III) For $M_1^1\nu < 0$, $(\bar{w}_1^{14})^{-1}M_2^4\nu > 0$ and $(\bar{w}_1^{14})^{-1}w_2^{14} > 0$, to guarantee $s_2 > 0$, (3.17) must have $0 < s_1 < \min\{s_{1L}, s_{1K}\}$, where $s_{1L} = \delta^{-1}M_2^1\nu + h.o.t.$ satisfying $L_1(s_{1L}) = 0$, and $s_{1K} = (-\bar{w}_1^{14}M_1^1\nu/M_2^4\nu)^{\lambda_1/\rho_2} + h.o.t.$ satisfying $K_1(s_{1K}) = 0$. Now we verify that $s_{1K} = (-\bar{w}_1^{14}M_1^1\nu/M_2^4\nu)^{\lambda_1/\rho_2} + h.o.t.$ is the small positive solution of $K_1(s_1) = 0$. Denote $t = s_1^{\rho_2/\lambda_1}$, and $\alpha = \lambda_1/\rho_2 + 1$, then it gives rise to

$$(\bar{w}_1^{14})^{-1}M_2^4\nu\delta^{-1}t + (\bar{w}_1^{14})^{-1}w_2^{14}t^\alpha + \delta^{-1}M_1^1\nu + h.o.t. = 0.$$

- As $(\bar{w}_1^{14})^{-1}M_2^4\nu > 0$ and $|M_1^1\nu|^{\frac{\alpha-1}{\alpha}} \ll |M_2^4\nu|$, it has a unique small positive solution $\bar{t} = -\bar{w}_1^{14}M_1^1\nu/M_2^4\nu + h.o.t.$, because $|(\bar{w}_1^{14})^{-1}w_2^{14}t^\alpha| = |(\bar{w}_1^{14})^{\alpha-1}w_2^{14}| \times |M_1^1\nu/M_2^4\nu|^\alpha \ll \delta^{-1}M_1^1\nu$. Thus, $K_1(s_1) = 0$ has a unique small positive solution $s_{1K} = (-\bar{w}_1^{14}M_1^1\nu/M_2^4\nu)^{\lambda_1/\rho_2} + h.o.t.$. Next we want to find a positive solution s_1 of $L_1(s_1) = K_1(s_1)$ satisfying $0 < s_1 < \min\{s_{1L}, s_{1K}\} \ll 1$. Because of $L_1'(s_1) < 0$, $K_1'(s_1) < 0$ and $|L_1'(s_1)| \gg |K_1'(s_1)|$ (as $\nu \rightarrow 0$), when $L_1(0) < K_1(0)$, $L_1(s_1) = K_1(s_1)$ has no positive solution; when $L_1(0) = K_1(0)$, $L_1(s_1) = K_1(s_1)$ has a solution $s_1 = 0$; when $L_1(0) > K_1(0)$, $L_1(s_1) = K_1(s_1)$ has a unique small positive solution as $s_{1L} \leq s_{1K}$, and no positive solution as $s_{1L} > s_{1K}$.
- (IV) For $M_1^1\nu > 0$, $(\bar{w}_1^{14})^{-1}M_2^4\nu < 0$ and $(\bar{w}_1^{14})^{-1}w_2^{14} < 0$, the first equation of (3.17) shows that $s_1 > 0$ as $s_2 > 0$, so we can eliminate s_1 and reduce (3.17) to $L_2(s_2) = K_2(s_2)$, where

$$\begin{aligned} L_2(s_2) &= s_2 + \delta^{-1}M_1^1\nu + h.o.t., \\ K_2(s_2) &= -(\bar{w}_1^{14})^{-1}M_2^4\nu\delta^{-1}((w_2^{12})^{-1}s_2^{\rho_1/\lambda_1} + \delta^{-1}M_2^1\nu)^{\rho_2/\lambda_1} \\ &\quad - (\bar{w}_1^{14})^{-1}w_2^{14}((w_2^{12})^{-1}s_2^{\rho_1/\lambda_1} + \delta^{-1}M_2^1\nu)^{\rho_2/\lambda_1+1} + h.o.t., \\ L_2(0) &= \delta^{-1}M_1^1\nu + h.o.t., \\ K_2(0) &= h(\nu). \end{aligned}$$

The proof of the remaining conclusion is similar to case (II).

- (V) For $w_2^{14}M_2^4\nu < 0$, $K_1'(s_1) = 0$ has a solution $s_1 = -\frac{\rho_2 M_2^4\nu}{(\rho_2 + \lambda_1)\delta w_2^{14}} + h.o.t. \triangleq \bar{s}_1$. Note

$$\begin{aligned} K_1(\bar{s}_1) &= -\delta^{-1}M_1^1\nu - \frac{\lambda_1 M_2^4\nu}{(\rho_2 + \lambda_1)\delta \bar{w}_1^{14}} \left(-\frac{\rho_2 M_2^4\nu}{(\rho_2 + \lambda_1)\delta w_2^{14}}\right)^{\rho_2/\lambda_1} + h.o.t., \\ K_1''(\bar{s}_1) &= \frac{\rho_2 M_2^4\nu}{\lambda_1 \delta \bar{w}_1^{14}} \left(-\frac{\rho_2 M_2^4\nu}{(\rho_2 + \lambda_1)\delta w_2^{14}}\right)^{\rho_2/\lambda_1-2} + h.o.t.. \end{aligned}$$

We can rewrite $K_1(s_1)$ as

$$K_1(s_1) = K_1(\bar{s}_1) + \frac{1}{2}K_1''(\bar{s}_1)(s_1 - \bar{s}_1)^2 + h.o.t.,$$

i.e.,

$$K_1(s_1) = D(\nu)s_1^2 + E(\nu)s_1 + F(\nu) + h.o.t..$$

Equation $K_1(s_1) = 0$ has two roots $s_{1k\pm}(\nu)$ given by

$$s_{1k\pm}(\nu) = \frac{-E(\nu) \pm \sqrt{E^2(\nu) - 4D(\nu)F(\nu)}}{2D(\nu)} + h.o.t.,$$

as $\Delta(\nu) > 0$. And $L_1(s_1) = 0$ has a positive root $s_{1L} = \delta^{-1}M_2^1\nu + h.o.t.$.

- (i) If $D(\nu) > 0$ and $F(\nu) < 0$, then $K_1(s_1) = 0$ has a unique positive root $s_{1K_+}(\nu)$. If $s_{1K_+}(\nu) < s_{1L}$, there is exactly a small positive root of $K_1(s_1) = L_1(s_1)$ on $(s_{1K_+}(\nu), s_{1L})$ and no positive root on $(0, s_{1K_+}(\nu))$. If $s_{1K_+}(\nu) = s_{1L}$, we have $s_2 = 0$. If $s_{1K_+}(\nu) > s_{1L}$, then $K_1(s_1) = L_1(s_1)$ has no positive root (see Figure 3).

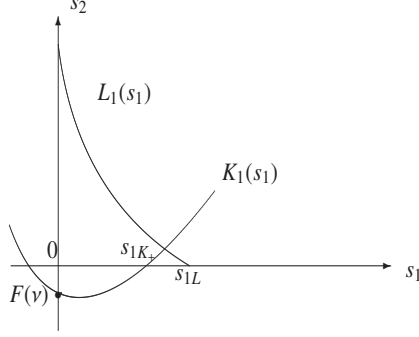


Figure 3. $D(\nu) > 0$ and $F(\nu) < 0$

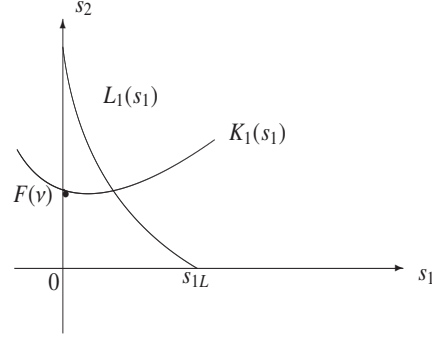


Figure 4. $D(\nu) > 0$ and $\Delta(\nu) < 0$

- (ii) If $D(\nu) > 0$ and $\Delta(\nu) < 0$, then $K_1(s_1) = 0$ has no positive root. If $F(\nu) < (-\delta^{-1}w_2^{12}M_2^1\nu)^{\lambda_1/\rho_1} + h.o.t.$, then $K_1(s_1) = L_1(s_1)$ has a unique small positive root. If $F(\nu) = (-\delta^{-1}w_2^{12}M_2^1\nu)^{\lambda_1/\rho_1} + h.o.t.$, we have $s_1 = 0$. If $F(\nu) > (-\delta^{-1}w_2^{12}M_2^1\nu)^{\lambda_1/\rho_1} + h.o.t.$, then $K_1(s_1) = L_1(s_1)$ has no positive root (see Figure 4).
- (iii) If $D(\nu) > 0$, $F(\nu) > 0$ and $\Delta(\nu) > 0$, then $K_1(s_1) = 0$ has two small positive roots $s_{1K_-}(\nu)$ and $s_{1K_+}(\nu)$. If $F(\nu) > (-\delta^{-1}w_2^{12}M_2^1\nu)^{\lambda_1/\rho_1} + h.o.t.$, then $K_1(s_1) = L_1(s_1)$ has no positive root. If $F(\nu) = (-\delta^{-1}w_2^{12}M_2^1\nu)^{\lambda_1/\rho_1} + h.o.t.$, we have $s_1 = 0$. When $0 < F(\nu) < (-\delta^{-1}w_2^{12}M_2^1\nu)^{\lambda_1/\rho_1} + h.o.t.$, if $s_{1k_-}(\nu) > \delta^{-1}M_2^1\nu + h.o.t.$ ($s_{1k_+}(\nu) < \delta^{-1}M_2^1\nu + h.o.t.$), then $K_1(s_1) = L_1(s_1)$ has a unique small positive root; if $s_{1k_-}(\nu) < \delta^{-1}M_2^1\nu + h.o.t. < s_{1k_+}(\nu)$, then $K_1(s_1) = L_1(s_1)$ has no positive root; if $s_{1k_-}(\nu) = \delta^{-1}M_2^1\nu + h.o.t.$ ($s_{1k_+}(\nu) = \delta^{-1}M_2^1\nu + h.o.t.$), we have $s_2 = 0$ (see Figure 5).
- (iv) If $D(\nu) < 0$ and $\Delta(\nu) < 0$, then $K_1(s_1) = L_1(s_1)$ has no positive root (see Figure 6).
- (v) If $D(\nu) < 0$, $F(\nu) < 0$ and $\Delta(\nu) > 0$, then $K_1(s_1) = 0$ has two small positive roots $s_{1K_-}(\nu)$ and $s_{1K_+}(\nu)$. If $s_{1k_-}(\nu) > \delta^{-1}M_2^1\nu + h.o.t.$ ($s_{1k_+}(\nu) < \delta^{-1}M_2^1\nu + h.o.t.$), then $K_1(s_1) = L_1(s_1)$ has no positive root. If $s_{1k_-}(\nu) = \delta^{-1}M_2^1\nu + h.o.t.$ ($s_{1k_+}(\nu) = \delta^{-1}M_2^1\nu + h.o.t.$), we have $s_2 = 0$. If $s_{1k_-}(\nu) < \delta^{-1}M_2^1\nu + h.o.t. < s_{1k_+}(\nu)$, then $K_1(s_1) = L_1(s_1)$ has a unique small positive root (see Figure 7).
- (vi) If $D(\nu) < 0$ and $F(\nu) > 0$, then $K_1(s_1) = 0$ has a unique positive root $s_{1K_+}(\nu)$. If $F(\nu) > (-\delta^{-1}w_2^{12}M_2^1\nu)^{\lambda_1/\rho_1} + h.o.t.$, then $K_1(s_1) = L_1(s_1)$ has no positive root. If $F(\nu) = (-\delta^{-1}w_2^{12}M_2^1\nu)^{\lambda_1/\rho_1} + h.o.t.$, we have $s_1 = 0$. When $0 < F(\nu) < (-\delta^{-1}w_2^{12}M_2^1\nu)^{\lambda_1/\rho_1} + h.o.t.$, if $s_{1k_+}(\nu) > \delta^{-1}M_2^1\nu + h.o.t.$, then $K_1(s_1) = L_1(s_1)$ has a unique

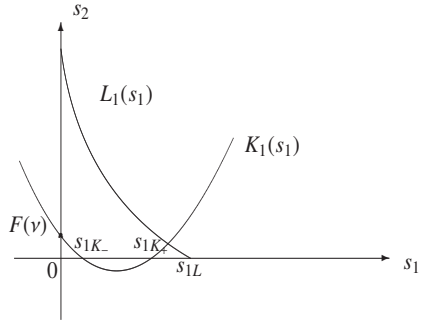


Figure 5. $D(\nu) > 0$, $F(\nu) > 0$ and $\Delta(\nu) > 0$

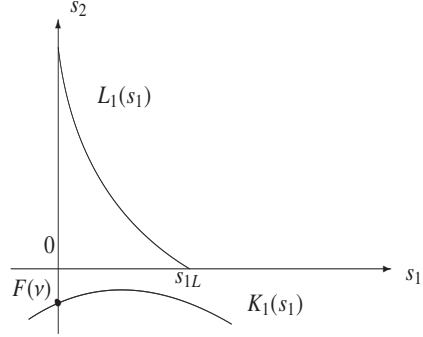


Figure 6. $D(\nu) < 0$ and $\Delta(\nu) < 0$

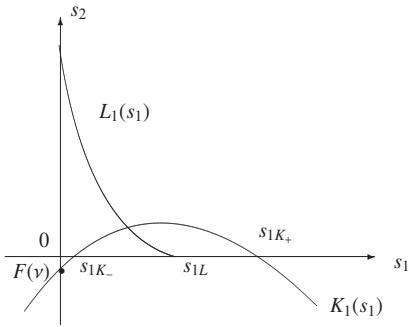


Figure 7. $D(\nu) < 0$, $F(\nu) < 0$ and $\Delta(\nu) > 0$

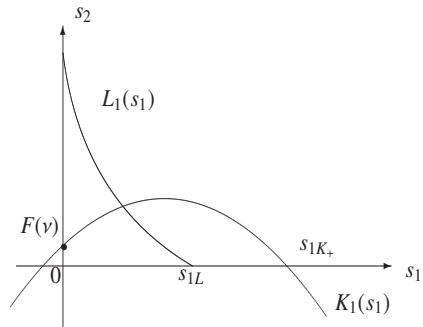


Figure 8. $D(\nu) < 0$ and $F(\nu) > 0$

small positive root; if $s_{1k_+}(\nu) = \delta^{-1}M_2^1\nu + h.o.t.$, we have $s_2 = 0$; if $s_{1k_+}(\nu) < \delta^{-1}M_2^1\nu + h.o.t.$, then $K_1(s_1) = L_1(s_1)$ has no positive root (see Figure 8).

5. Application

Take into account the C^r system

$$\dot{z} = f(z) + g(z, \nu), \quad (5.1)$$

with its unperturbed system

$$\dot{z} = f(z), \quad (5.2)$$

where $z = (z_1, z_2, z_3, z_4)$, $\nu = (\nu_1, \nu_2, \nu_3)$, $g(z, 0) = 0$,

$$f(z) = \begin{pmatrix} 3z_1^2 + 2z_2^2 + 4z_1 \\ (z_1 - 3)z_2 \\ z_3(1 - z_1) \\ z_4(z_1 - 6) \end{pmatrix}, \quad g(z, \nu) = \begin{pmatrix} z_1^2\nu_1 \\ z_1^{1/2}z_1\nu_2 \\ z_1\nu_1 \\ z_1^{1/2}\nu_3 \end{pmatrix}.$$

When $\nu = 0$, system (5.1) has a equilibrium $P = (0, 0, 0, 0)$ and a double homoclinic cycle $\Gamma = \Gamma_1 \cup \Gamma_2$, which is expressed by $\Gamma_i = \{z = r_i(t), t \in R\}$, $i = 1, 2$. Here

$$r_1(t) = \left(\frac{1}{e^t + Ce^{-t}}, 0, 0, 0\right), \quad r_2(t) = \left(\frac{1}{e^{2t} + Ce^{-2t}}, 0, 0, 0\right) \quad (5.3)$$

satisfying $r_1(+\infty) = r_1(-\infty) = r_2(+\infty) = r_2(-\infty) = P$.

Set

$$\frac{1}{e^t + Ce^{-t}} = \delta, \quad t \rightarrow \pm\infty,$$

where $\delta > 0$ sufficiently small, then we have $C = 1$. Choose $2\sigma = 1/\delta + (1/\delta^2 - 4)^{1/2}$, therefore, $T = \ln \sigma$.

Due to

$$Df(z) = \begin{pmatrix} 6z_1 + 4 & 4z_2 & 0 & 0 \\ z_2 & z_1 - 3 & 0 & 0 \\ -z_3 & 0 & 1 - z_1 & 0 \\ z_4 & 0 & 0 & z_1 - 6 \end{pmatrix},$$

we have $Df(P) = \text{diag}(4, -3, 1, -6)$.

Now, we consider the linear variational system of unperturbed system (5.2) and its adjoint system

$$\dot{z} = Df(r_i(t))z, \quad (5.4)_i$$

$$\dot{z} = -(Df(r_i(t)))^* z, \quad (5.5)_i$$

where $i = 1, 2$,

$$Df(r_1(t)) = \text{diag}\left(\frac{6}{e^t + e^{-t}} + 4, \frac{1}{e^t + e^{-t}} - 3, 1 - \frac{1}{e^t + e^{-t}}, \frac{1}{e^t + e^{-t}} - 6\right).$$

One fundamental solution matrix for (5.4)₁ is

$$\hat{Z}_1(t) = \text{diag}(C_1 e^{\int_0^t (\frac{6}{e^s + e^{-s}} + 4) ds}, C_2 e^{\int_0^t (\frac{1}{e^s + e^{-s}} - 3) ds}, C_3 e^{\int_0^t (1 - \frac{1}{e^s + e^{-s}}) ds}, C_4 e^{\int_0^t (\frac{1}{e^s + e^{-s}} - 6) ds}).$$

Correspondingly, one has

$$\begin{aligned} \hat{\Psi}_1(t) = (\hat{Z}_1^{-1}(t))^* &= \text{diag}(C_1^{-1} e^{-\int_0^t (\frac{6}{e^s + e^{-s}} + 4) ds}, C_2^{-1} e^{-\int_0^t (\frac{1}{e^s + e^{-s}} - 3) ds}, \\ &C_3^{-1} e^{-\int_0^t (1 - \frac{1}{e^s + e^{-s}}) ds}, C_4^{-1} e^{-\int_0^t (\frac{1}{e^s + e^{-s}} - 6) ds}), \end{aligned}$$

where $C_j, j = 1, \dots, 4$ are constant to be determined.

Furthermore, we should perform the coordinates transformation by

$$z_1 \rightarrow u, z_2 \rightarrow y, z_3 \rightarrow x, z_4 \rightarrow v$$

in the small neighborhood of P , so as to match well with the local moving frame.

Thus, we obtain

$$Z_1(t) = \begin{pmatrix} C_3 e^{\int_0^t (1 - \frac{1}{e^s + e^{-s}}) ds} & 0 & 0 & 0 \\ 0 & C_2 e^{\int_0^t (\frac{1}{e^s + e^{-s}} - 3) ds} & 0 & 0 \\ 0 & 0 & C_1 e^{\int_0^t (\frac{6}{e^s + e^{-s}} + 4) ds} & 0 \\ 0 & 0 & 0 & C_4 e^{\int_0^t (\frac{1}{e^s + e^{-s}} - 6) ds} \end{pmatrix},$$

for $t \in [T_1, +\infty)$,

and

$$Z_1(t) = \begin{pmatrix} 0 & C_3 e^{\int_0^t (1 - \frac{1}{e^s + e^{-s}}) ds} & 0 & 0 \\ 0 & 0 & 0 & C_2 e^{\int_0^t (\frac{1}{e^s + e^{-s}} - 3) ds} \\ 0 & 0 & C_1 e^{\int_0^t (\frac{6}{e^s + e^{-s}} + 4) ds} & 0 \\ C_4 e^{\int_0^t (\frac{1}{e^s + e^{-s}} - 6) ds} & 0 & 0 & 0 \end{pmatrix},$$

for $t \in (-\infty, -T_1]$.

Since

$$Z_1(-T_1) = \begin{pmatrix} w_1^{11} & w_1^{21} & 0 & w_1^{41} \\ 0 & 0 & 0 & w_1^{42} \\ w_1^{13} & 0 & 1 & w_1^{43} \\ \bar{w}_1^{14} & 0 & 0 & w_1^{44} \end{pmatrix}, \quad Z_1(T_1) = \begin{pmatrix} 1 & 0 & w_1^{31} & 0 \\ 0 & 1 & w_1^{32} & 0 \\ 0 & 0 & w_1^{33} & 0 \\ w_1^{14} & 0 & w_1^{34} & 1 \end{pmatrix},$$

with

$$\begin{aligned} C_1 &= \frac{\sigma^4 e^{6 \arctan 1}}{e^{6 \arctan(1/\sigma)}}, & C_2 &= \frac{\sigma^3 e^{\arctan 1}}{e^{\arctan \sigma}}, \\ C_3 &= \frac{e^{\arctan \sigma}}{\sigma e^{\arctan 1}}, & C_4 &= \frac{\sigma^6 e^{\arctan 1}}{e^{\arctan \sigma}}, \end{aligned}$$

and

$$\begin{aligned} w_1^{11} &= w_1^{13} = w_1^{41} = w_1^{43} = w_1^{44} = w_1^{14} = w_1^{31} = w_1^{32} = w_1^{34} = 0, \\ \bar{w}_1^{14} &= \frac{\sigma^{12} e^{\arctan(1/\sigma)}}{e^{6 \arctan \sigma}}, & w_1^{21} &= \frac{e^{\arctan \sigma}}{\sigma^2 e^{\arctan(1/\sigma)}}, \\ w_1^{42} &= \frac{\sigma^6 e^{\arctan(1/\sigma)}}{e^{6 \arctan \sigma}}, & w_1^{33} &= \frac{\sigma^8 e^{\arctan \sigma}}{e^{\arctan(1/\sigma)}}, \end{aligned}$$

note that

$$\Psi_1(t) = \begin{pmatrix} 0 & C_3^{-1} e^{-\int_0^t (1 - \frac{1}{e^s + e^{-s}}) ds} & 0 & 0 \\ 0 & 0 & 0 & C_2^{-1} e^{-\int_0^t (\frac{1}{e^s + e^{-s}} - 3) ds} \\ 0 & 0 & C_1^{-1} e^{-\int_0^t (\frac{6}{e^s + e^{-s}} + 4) ds} & 0 \\ C_4^{-1} e^{-\int_0^t (\frac{1}{e^s + e^{-s}} - 6) ds} & 0 & 0 & 0 \end{pmatrix},$$

for $t \in \mathbb{R}$.

In the following, we can calculate

$$\begin{aligned} M_1^1 &= (0, 0, \frac{1}{C_4} \int_{-\infty}^{+\infty} (\frac{1}{e^t + e^{-t}})^{1/2} \frac{1}{e^{\arctan e^t - \arctan 1 - 6t}} dt), \\ M_1^3 &= (\frac{1}{C_1} \int_{-\infty}^{+\infty} (\frac{1}{e^t + e^{-t}}) \frac{1}{6e^{\arctan e^t - \arctan 1 + 4t}} dt, 0, 0), \\ M_1^4 &= (0, \frac{1}{C_2} \int_{-\infty}^{+\infty} (\frac{1}{e^t + e^{-t}})^{3/2} \frac{1}{e^{\arctan e^t - \arctan 1 - 3t}} dt, 0). \end{aligned}$$

So we verify that this example is consistent with our work.

6. Conclusion

In this paper, we study codimension 3 bifurcations of double homoclinic loops with W^s inclination flip of Γ_1 in the case of $\rho_1 > \lambda_1$. We assume that $r \geq 3Q$ and $D^N f(0) = 0$ for $N = 0, 1$. It satisfies the Sternberg condition of order Q , so system (2.1) is uniformly C^K linearizable according to [12]. Then we can reduce system (2.1) to a simpler normal form without higher order terms. By setting up local active coordinates approach in some tubular neighborhood of unperturbed double homoclinic loops and using the fundamental solution matrix of the linear variational system in regard to the elementary cycle, we construct Poincaré return map and the successor functions and obtain bifurcation equations.

In the forementioned bifurcation analysis, we prove the possible bifurcations of the double homoclinic loops connected with a hyperbolic critical points in a four dimensional space. It is interesting to find the existence and incoexistence of the double homoclinic loops, large 1-homoclinic orbit and large 1-periodic orbit in

Theorem 4.1 and the sufficient conditions for Γ_i or Γ being persistent in Theorem 4.2. We also discuss the existence or nonexistence of large 1-periodic orbits and large 1-homoclinic orbits near Γ in Theorem 4.3. It is worthy to be mentioned that the restriction on the dimension is not essential, and the method used in this paper can be extended to higher dimensional systems without any difficulty under the same hypotheses. But the difficulty of these problems will increase with adding codimension of the double homoclinic loops. Finally, we will be interesting to study some biological and epidemiological models by applying the results obtained in this work. We leave these for future research.

References

- [1] D. Blázquez-Sanz and K. Yagasaki, *Analytic and algebraic conditions for bifurcations of homoclinic orbits I: Saddle equilibria*, J. Differ. Equations, 2012, 253(11), 2916-2950.
- [2] S.N. Chow, B. Deng and B. Fiedler, *Homoclinic bifurcation at resonant eigenvalues*, J. Dyn. Differ. Equ., 1990, 2(2), 177-244.
- [3] S.N. Chow and X.B. Lin, *Bifurcation of a homoclinic orbit with a saddle node equilibrium*, Differ. Integral Equ., 1990, 3(3), 435-466.
- [4] B. Deng, *Homoclinic twisting bifurcations and cusp horseshoe maps*, J. Dyn. Differ. Equ., 1993, 5(3), 417-467.
- [5] J. Gruendler, *Homoclinic solutions for autonomous dynamical systems in arbitrary dimension*, SIAM J. Math. Anal., 1992, 23(3), 702-721.
- [6] J. Gruendler, *Homoclinic solutions for autonomous ordinary differential equations with nonautonomous perturbations*, J. Differ. Equations, 1995, 122(1), 1-26.
- [7] M.A. Han and P. Bi, *Bifurcation of limit cycles from a double homoclinic loop with a rough saddle*, Chinese Ann. Math. Ser. B, 2004, 25(2), 233-242.
- [8] M.A. Han and J. Chen, *On the number of limit cycles in double homoclinic bifurcations*, Sci. China Ser. A, 2000, 43(9), 914-928.
- [9] A.J. Homburg, *Global aspects of homoclinic bifurcations of vector fields*, Mem. Amer. Math. Soc., 1996, 121(578).
- [10] A.J. Homburg and J. Knobloch, *Multiple homoclinic orbits in conservative and reversible systems*, Trans. Amer. Math. Soc., 2006, 358(4), 1715-1740.
- [11] A.J. Homburg and B. Krauskopf, *Resonant homoclinic flip bifurcations*, J. Dyn. Differ. Equ., 2000, 12(4), 807-850.
- [12] G.R. Sell, *Smooth linearization near a fixed point*, Amer. J. Math., 1985, 107(5), 1035-1091.
- [13] Y.L. Jin and D.M. Zhu, *Bifurcation of rough heteroclinic loop with two saddle points*, Sci. China Ser. A, 2003, 46(4), 459-468.
- [14] M. Kisaka, H. Kokubu and H. Oka, *Bifurcations to n -homoclinic orbits and n -periodic orbits in vector fields*, J. Dyn. Differ. Equ., 1993, 5(2), 305-357.
- [15] H. Kokubu, M. Komuru and H. Oka, *Multiple homoclinic bifurcations from orbit flip I. Successive homoclinic doublings*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 1996, 6(5), 833-850.

- [16] J. Knobloch and T. Wagenknecht, *Homoclinic snaking near a heteroclinic cycle in reversible systems*, Phys. D, 2005, 206(1-2), 82-93.
- [17] X.B. Liu and D.M. Zhu, *Bifurcation of degenerate homoclinic orbits to saddle-center in reversible systems*, Chin. Ann. Math. Ser. B, 2008, 29(6), 575-584.
- [18] Q.Y. Lu, *Codimension 2 bifurcation of twisted double homoclinic loops*, Comput. Math. Appl., 2009, 57(7), 1127-1141.
- [19] Q.Y. Lu, G.F. Deng and H. Luo, *Codimension 3 bifurcation from orbit-flip homoclinic orbit of weak type*, Electron. J. Qual. Theory Differ. Equ., 2015, (71), 1-12.
- [20] C.A. Morales, M.J. Pacifico and B. San Martin, *Expanding Lorenz attractors through resonant double homoclinic loops*, SIAM J. Math. Anal., 2005, 36(6), 1836-1861.
- [21] C.A. Morales, M.J. Pacifico and B. San Martin, *Contracting Lorenz attractors through resonant double homoclinic loops*, SIAM J. Math. Anal., 2006, 38(1), 309-332.
- [22] B.E. Oldeman, B. Krauskopf and A.R. Champneys, *Numerical unfoldings of codimension-three resonant homoclinic flip bifurcations*, Nonlinearity, 2001, 14(3), 597-621.
- [23] C. Grotta Ragazzo, *On the stability of double homoclinic loops*, Comm. Math. Phys., 1997, 184(2), 251-272.
- [24] B. Sandstede, *Constructing dynamical systems having homoclinic bifurcation points of codimension two*, J. Dyn. Differ. Equ., 1997, 9(2), 269-288.
- [25] S. Schecter and C. Sourdis, *Heteroclinic orbits in slow-fast Hamiltonian systems with slow manifold bifurcations*, J. Dyn. Differ. Equ., 2010, 22(4), 629-655.
- [26] S.L. Shui and D.M. Zhu, *Codimension 3 nonresonant bifurcations of homoclinic orbits with two inclination flips*, Sci. China Ser. A, 2005, 48(2), 248-260.
- [27] S.L. Shui and D.M. Zhu, *Codimension 3 non-resonant bifurcations of rough heteroclinic loops with one orbit flip*, Chinese Ann. Math. B, 2006, 27(6), 657-674.
- [28] S.A. van Gils, M. Krupa and V. Tchistiakov, *Homoclinic twist bifurcation in a system of two coupled oscillators*, J. Dyn. Differ. Equ., 2000, 12(4), 733-806.
- [29] T. Wagenknecht, *Two-heteroclinic orbits emerging in the reversible homoclinic pitchfork bifurcation*, Nonlinearity, 2005, 18(2), 527-542.
- [30] S. Wiggins, *Global Bifurcations and Chaos: Analytical Methods*, Springer-Verlag, New York, 1988.
- [31] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer-Verlag, New York, 1990.
- [32] T.S. Zhang and D.M. Zhu, *Codimension 3 homoclinic bifurcation of orbit flip with resonant eigenvalues corresponding to the tangent directions*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 2004, 14(12), 4161-4175.
- [33] W.P. Zhang and D.M. Zhu, *Codimension 2 bifurcations of double homoclinic loops*, Chaos, Solitons and Fractals, 2009, 39(1), 295-303.

-
- [34] W.P. Zhang, D.M. Zhu and D. Liu, *Codimension 3 nontwisted double homoclinic loops bifurcations with resonant eigenvalues*, J. Dyn. Diff. Equat., 2008, 20(4), 893-908.
- [35] D.M. Zhu, *Problems in homoclinic bifurcation with higher dimensions*, Acta Math. Sinica (N.S.), 1998, 14(3), 341-352.
- [36] D.M. Zhu and Z.H. Xia, *Bifurcations of heteroclinic loops*, Sci. China Ser. A, 1998, 41(8), 837-848.