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A Consumption Behavior Model with Advertising And Word-of-Mouth Effect

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Abstract It is widely-held belief that people's consumption behavior is partly determined by advertising and word-of-mouth effect especially in monopolistic competitive market. Owing to this, we propose a mathematical model to interpret consumer behavior under the advertising and word-of-mouth effects, which is divided into continuous and discrete types for dynamic behavior analysis. Our research indicates the continuous model undergoes fold bifurcation, Hopf bifurcation, and degenerate fold-Hopf bifurcation; the discrete model undergoes flip bifurcation and Neimark-Sacker bifurcation. Moreover, bifurcation diagrams are given by using MATLAB to illustrate the model. Based on the theory of Hopf bifurcation or Flip bifurcation, the system undergoes supercritical Hopf bifurcation or Flip bifurcation under certain conditions. This lead to both advertising impacts periodically on consumer behavior, and a short-term reduction cause no effect the public¡s recognition of the brand, which can guide precision advertising investment.

Keywords The consumption behaviour model, Hopf bifurcation, fold bifurcation, Flip bifurcation, Neimark-Sacker bifurcation.

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1. Introduction

As the main way for consumers to understand product information, commercial advertising spreads new consumption concepts, supplies new consumption methods, and guides consumer demand. Advertisement also play an important role in four different market conditions, namely perfect competition, monopolistic competition, oligopoly and monopoly. Under the oligopoly competition market, for example, the international company Coca-Cola invest huge amounts of money in a large number of advertisements, creating fashion slogans all the time, planning creative advertisements, and leads young people's demand for Coca-Cola. As a corporation in monopolistic competition market, Apple company promoted the Spring Festival brand advertisement in China since 2015. We note in Figure 1 (Data are from Apple quarterly earnings *) that revenue in Greater China have increased significantly in

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*https://www.apple.com/cn/newsroom

2015 Q1, 2016 Q1, 2017 Q1 and 2018 Q1 (Q1 indicates the first quarter). This conveys that the effective advertising is able to raise the maximum price consumers are willing to pay, product sales and eventually to boost profits.

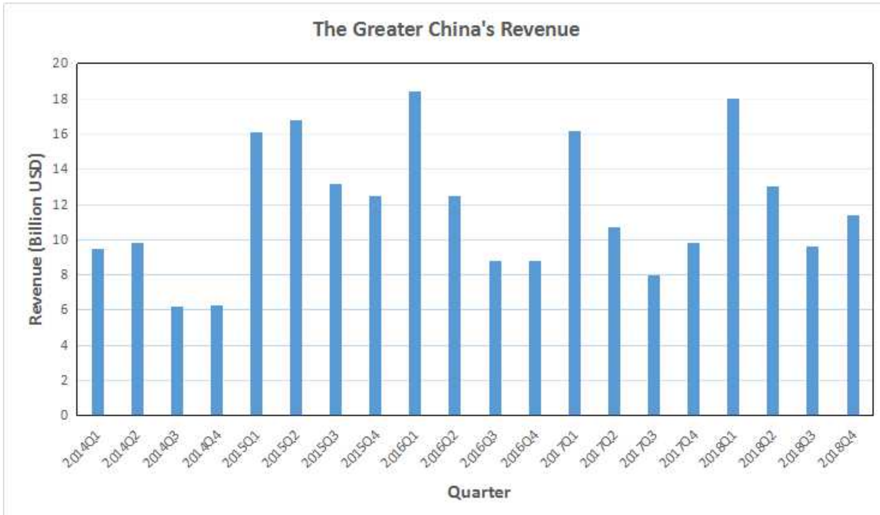

Figure 1. Apple's quarterly earnings report for Greater China.

Advertising is vital in our daily life. It mainly determines the impact on our thinking as well as on the attitude towards different and new products. An excellent and creative advertisement can disseminate, arouse people's desire to buy and thus realize the commodity value. While the impacts of advertising and sales promotion in revenue are often used interchangeably, advertising and sales promotion designate different parts in enterprise marketing strategy. Advertising has to do with building brand recognition and taking measures in order to secure long-term profitability, whereas sales promotions are short-term strategies infusing immediate revenue into a business by directly affecting the price of goods or services.

In one social science study [1], the negative and positive effect of word of mouth marketing on consumer purchasing behavior have been examined. The study is based on primary data collected from numerous households and university students from the area of Rawalpindi and Islamabad cities of Pakistan, the results have revealed that consumers tend to rely on word of mouth for the purchase of not only everyday items, but long-term goods as well. The person who have a great influence on the decision of consumers are close family, friends and acquaintances. Moreover, the research imputes the failure of revenue-poor companies mostly to adverse word of mouth. Viral Marketing is gaining rapid popularity among consumers as it is depicted in the results. Finally, a negative word of mouth travels faster and more widely spread than positive comments.

In view of the fact that advertising and word-of-mouth effects are dynamic economic phenomena, many studies have used the Innovation diffusion theory because of its great economic importance since its appearance in the previous century sixties. Numerous mathematical models have been established to market new products. Sales Promotions were widely discussed in many studies from both mathematical and economic standpoints. Many of these models aim to explain the spread of a new product among potential customers of the population, by considering the effects of oral speech, advertising, other forms of communication, etc.

In [4], Feichtinger divided the consumer groups into potential consumers and costumers so as to study how the effectiveness of advertising affects the spread of goods in consumer groups. Subsequently, Landa considered the attractiveness of marketing activities, improved the model of Feichtinger and analyzed its dynamic behavior [5]. In [17], Nicoleta considered that the advertising effect was delayed by economic, cultural, social and other factors; studied the advertising diffusion model with time lag. In [23], Yang and Zhang established a random diffusion model based on the effects of word of mouth and advertising on product diffusion. In [6], Paolo et al. used statistical correlation models to study consumer behavior and describe some methodological problems related to the implementation of discrete graphical models for market basket analysis data. In [10], Jiang and Ma studied the differential advertising model with internet sales promotion, and used bifurcation theory to analyze the conditions for the existence and stability of periodic solutions. Promotion have been envisioned as parameters that can change the stability of differential advertising model by flipping bifurcations and lead to chaos. In practical applications, when total cost stays constant, corporation can make optimal promotion strategies to gain most revenue by adjusting the value of promotion parameters, and ultimately to maximize profits.

In the market, advertising influence does not increase linearly with time. So how can we describe advertising effect in mathematical language? In [18], Sun et al. studied the factors that affect people's coming to China by observing the spending of the United States and South Korea, GDP and other information. As a way of information dissemination, advertising can deliver information to the public at the same time. Wang and others [22] put forward Logistic diffusion model considering the dissemination of information in time and space dimension, and used the real dataset collected from Digg social news site, the results highly consistent with the actual data. For further study of the dynamic behavior of multi-information propagation, Ren and Yu [14] constructed a two-dimensional discrete model to study the interference of two pieces of information. They partition the model into three cases: anti-transformation, suppression and mutual aid, respectively, studied 1:2, 1:3, 1:4 resonance, and given two control strategies. In [16], Ren et al. studied multiple information diffusion models with free boundary conditions, and gave the situation that information is gradually developed under several different conditions, and countless examples are given to demonstrate the initial extended region and the extended ability to the free boundary.

Based on their work, we consider that advertising and word-of-mouth effects in the market affect sales of goods at the same time, meanwhile propose a three-dimensional consumer behavior model. Based on bifurcation Theory according to Reference [9, 13, 15, 19, 20], this paper analyzes characterize the dynamic behaviors from two types of continuous and discrete types, also provides theoretical guidance for merchants to make decisions in response to market changes.

This paper is organized as follows. In Section 2 and Section 3, we analyze the dynamical behavior of continuous consumption behavior model and discrete consumption behavior model respectively. Finally, we conclude the paper with a brief discussion in Section 4.

2. Dynamic analysis of the continuous consumption behavior model

In this paper, a consumption behavior model is studied under the background of information dissemination and economic phenomena.

2.1. Our model

Based on Feichtinger's advertising diffusion model [4],

$$\begin{cases} \dot{x_1} = k - \alpha x_1 x_2^2 + \beta x_2, \\ \dot{x_2} = \alpha x_1 x_2^2 - (\beta + \epsilon) x_2, \end{cases}$$

we classify the consumer groups into two categories: potential consumers and consumers, and assume that word-of-mouth effect is proportional to the number of consumers. In addition, the dissemination of advertising is regarded as information dissemination. Based on the logistic diffusion information model studied by Wang [22],

$$\begin{cases} \frac{\partial I}{\partial t} = d\frac{\partial^2 I}{\partial t^2} + rI(1 - \frac{I}{K}), \\ I(x,l) = \phi(x), l < I < L, \\ \frac{\partial I}{\partial t}(l,t) = \frac{\partial I}{\partial t}(L,t) = 0, t > 1, \end{cases}$$

the logistic curve is used to describe the change of advertising influence over time. From this, we build the model

$$\begin{cases} \dot{x_1} = k + bx_1x_3 - ax_1x_2^2 - cx_1x_3(1 - x_3) + \beta x_2, \\ \dot{x_2} = ax_1x_2^2 + cx_1x_3(1 - x_3) - (\beta + \epsilon)x_2, \\ \dot{x_3} = dx_3(1 - \frac{x_3}{e}), \end{cases} \tag{2.1}$$

$x_1(t), x_2(t)$ and $x_3(t)$ represent the number of potential consumers, the number of consumers and the impact of advertising over time, respectively. The parameters in the model are shown in Table 1, k, ϵ, a, b, e are positive numbers, β, c, d are non-negative numbers.

Table 1. Coefficient of the model.

k:	natural inflow rate of potential consumers in the market;
ϵ:	current customers leave the market forever;
β:	the ratio of consumers converting to potential consumers again;
a:	the impact rate of Word-of-Mouth on potential consumers;
b:	the impact rate of advertising on potential Consumers;
c:	the ratio of potential consumers buying products due to advertising;
d:	one of logistic curve control parameters;
e:	one of logistic curve control parameters;

In order to facilitate the analysis of its dynamical behavior, we obtain the topological equivalent system of the system through a series of transformations. Let $n = (\beta + \epsilon), m = (b - c)$, it's transformed into

$$\begin{cases} \dot{x_1} = k + mx_1x_3 - ax_1x_2^2 + cx_1x_3^2 + \beta x_2, \\ \dot{x_2} = ax_1x_2^2 + cx_1x_3(1 - x_3) - nx_2, \\ \dot{x_3} = dx_3(1 - \frac{x_3}{e}). \end{cases}$$

The system variables can be represented linearly as follows

$$x = \frac{ak}{n\epsilon}x_1,\ y = \frac{\epsilon}{k}x_2,\ z = \frac{\epsilon}{k}x_3,\ \tau = nt,$$

we obtain

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{ak^2}{n\epsilon^2}[1 - xy^2 + \frac{m\epsilon}{ak}xz + \frac{c}{a}xz^2 + \frac{\beta}{n}(y - 1)], \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} = xy^2 - y - \frac{c}{a}xz^2 + \frac{c\epsilon}{ak}xz, \\ \frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{dk}{ne\epsilon}z(\frac{e\epsilon}{k} - z). \end{cases} \tag{2.2}$$

By placing $m_1 = \frac{ak^2}{n\epsilon^2}, m_2 = \frac{m\epsilon}{ak}, m_3 = \frac{c}{a}, m_4 = \frac{\beta}{n}, m_5 = \frac{c\epsilon}{ak}, m_6 = \frac{kd}{ne\epsilon}, m_7 = \frac{e\epsilon}{k}$, it becomes

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}\tau} = m_1[1 - xy^2 + m_2xz + m_3xz^2 + m_4(y - 1)], \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} = xy^2 - y - m_3xz^2 + m_5xz, \\ \frac{\mathrm{d}z}{\mathrm{d}\tau} = m_6z(m_7 - z), \end{cases} \tag{2.3}$$

$m_i > 0\ (i = 1, 7),\ m_j \geq 0\ (j = 3, 5, 6),\ 0 \leq m_4 < 1$ and $m_2 + m_5 > 0$.

Due to the new time transformation, systems (2.1) and (2.3) are topologically equivalent. So we are going to study system (2.3) instead of (2.1).

2.2. Stability of equilibriums

We can easily find that, system (2.3) has a trivial equilibrium $E_0(1, 1, 0)$. For any non-trivial equilibrium point $E_1(x_*, y_*, z_*)$, it lies on

$$\begin{cases} x_* = \frac{1-m_4}{m_7(m_2+m_5)}(y_* - 1), \\ y_*^3 - y_*^2 + ry_* + h = 0, \\ z_* = m_7, \end{cases}$$

here

$$\begin{cases} r = \frac{m_7}{1-m_4}[m_3m_7(m_4 - 1) - m_2 - m_4m_5] = -\frac{e\epsilon(mn+c(\beta+e\epsilon))}{ak^2}, \\ h = m_7(m_3m_7 - m_5) = \frac{c(e-1)e\epsilon^2}{ak^2}, \\ r + h = -\frac{m_7}{1-m_4}(m_2 + m_5) = -\frac{e\epsilon(mn+c(\beta+\epsilon))}{ak^2} < 0. \end{cases}$$

Since $x \geq 0, y \geq 0$, we only consider the roots of equation $y^3 - y^2 + ry + h = 0$, which are greater than or equal to 1.

Let $u = y - 1$, $y = u + 1$, hence

$$u^3 + 2u^2 + (r + 1)u + (r + h) = 0. \tag{2.4}$$

According to the distribution of the roots of the cubic equation, we can get the following conclusions

Lemma 2.1. *The non-negative root distribution of the equation* (2.4)

1. $r + h < 0$, *The equation* (2.4) *has a unique positive root;*
2. $r + h = 0, r \geq -1$, *The equation* (2.4) *has a unique zero root;*
3. $r + h = 0, r < -1$, *The equation* (2.4) *has one zero root and one positive root;*
4. $r + h > 0, r < -1, h = \frac{2p^3+3p^2-1}{27}(p = \sqrt{1-3r})$, *The equation* (2.4) *has a unique positive root;*
5. $r + h = 0, r < -1, h < \frac{2p^3+3p^2-1}{27}(p = \sqrt{1-3r})$, *The equation* (2.4) *has two positive roots.*

According to the lemma (2.1), $r+h < 0$, the equation (2.4) has a unique positive root, and the system (2.3) has a unique non-trivial equilibrium point $E_1(x_*, y_*, z_*)$.

Next, we will discuss the stability of both equilibriums separately.

The Jacobian matrix for the system (2.3) at any point $E(x, y, z)$ can be expressed as

$$J(E) = \begin{pmatrix} m_1(-y^2 + m_2 z + m_3 z^2) & m_1(-2xy + m_4) & m_1(m_2 x + 2m_3 xz) \\ y^2 - m_3 z^2 + m_5 z & 2xy - 1 & -2m_3 xz + m_5 x \\ 0 & 0 & m_6(m_7 - 2z) \end{pmatrix}.$$

Then, we get the Jacobian matrix at the equilibrium $E_0(1, 1, 0)$ as

$$J(E_0) = \begin{pmatrix} -m_1 & m_1(-2 + m_4) & m_1 m_2 \\ 1 & 1 & m_5 \\ 0 & 0 & m_6 m_7 \end{pmatrix}.$$

The characteristic equation of $J(E_0)$ has the form

$$(\lambda - m_6 m_7)\Big(\lambda^2 + (m_1 - 1)\lambda + m_1(1 - m_4)\Big) = 0. \tag{2.5}$$

By the straightforward calculation, we obtain $\lambda_1 = m_6 m_7$, and the other two eigenvalues $\lambda_{2,3}$ of $J(E_0)$ satisfy the following equation

$$\lambda^2 + (m_1 - 1)\lambda + m_1(1 - m_4) = 0.$$

With respect to the equation (2.5), we can draw the following conclusions.

Lemma 2.2. *The eigenvalues $\lambda_{2,3}$ of $J(E_0)$ satisfy the following conditions*

1. $Re\lambda_2 > 0, Re\lambda_3 > 0$ *if* $m_1 < 1,\ m_4 < 1$;
2. $Re\lambda_2 < 0, Re\lambda_3 < 0$ *if* $m_1 > 1,\ m_4 < 1$;
3. $Re\lambda_i = 0, Im\lambda_i \neq 0 (i = 2, 3)$ *if* $m_1 = 1,\ m_4 < 1$;

Due to lemma (2.2), we obtain the next results

Theorem 2.1. *The trivial equilibrium point $E_0(1, 1, 0)$ is*

- *an unstable if* $m_6 > 0,\ m_1 < 1$;

- *a hyperbolic saddle if* $m_6 > 0,\ m_1 > 1$*;*
- *a non-hyperbolic point if the parameter satisfies one of the following conditions*
  1. $m_6 = 0,\ m_1 \neq 1$*;*
  2. $m_6 > 0,\ m_1 = 1$*;*
  3. $m_6 = 0,\ m_1 = 1$.

In order to facilitate the study of the stability of (2.3) at the non-trivial equilibrium $E_1(x_*, y_*, z_*)$, we bring E_1 to the origin by using the following transformation

$$u = x - x_*, v = y - y_*, w = z - m_7,$$

which yield that

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}\tau} = a_{11}u + a_{12}v + a_{13}w + m_1[-2y_*uv + (m_2 + 2m_3m_7)uw - \kappa(y_* - 1)v^2 \\ \qquad + \kappa m_3(y_* - 1)w^2 - uv^2 + m_3uw^2], \\ \dfrac{\mathrm{d}v}{\mathrm{d}\tau} = a_{21}u + a_{22}v + a_{23}w + 2y_*uv + (m_5 - 2m_3m_7)uw + \kappa(y_* - 1)v^2 \\ \qquad - \kappa m_3(y_* - 1)w^2 + uv^2 - m_3uw^2, \\ \dfrac{\mathrm{d}w}{\mathrm{d}\tau} = - m_6w(w + m_7), \end{cases} \tag{2.6}$$

where

$$\kappa = \frac{1 - m_4}{m_7(m_2 + m_5)},$$

$$\begin{aligned} a_{11} &= m_1(-y_*^2 + m_2m_7 + m_3m_7^2), & a_{12} &= m_1(-2\kappa y_*^2 + 2ky_* + m_4), \\ a_{13} &= m_1\kappa(m_2 + 2m_3m_7)(y_* - 1), & a_{21} &= y_*^2 + m_5m_7 - m_3m_7^2, \\ a_{22} &= 2\kappa y_*(y_* - 1) - 1, & a_{23} &= \kappa(m_5 - 2m_3m_7)(y_* - 1). \end{aligned}$$

By straightforward calculation, the characteristic equation of E_1 is given as

$$f(\lambda) = (\lambda + m_6m_7)(\lambda^2 - a_1\lambda + a_2) = 0, \tag{2.7}$$

where

$$\begin{aligned} a_1 &= \Big(\frac{2(1 - m_4)}{m_7(m_2 + m_5)} - m_1\Big)y_*^2 - \frac{2(1 - m_4)}{m_7(m_2 + m_5)}y_* + m_1m_7(m_2 + m_3m_7) - 1, \\ a_2 &= 3m_1(1 - m_4)y_*^2 - 2m_1(1 - m_4)y_* - m_1m_7\big(m_2 + m_4m_5 + m_3m_7(1 - m_4)\big). \end{aligned}$$

One can verify an eigenvalue is $\lambda_1 = -m_6m_7$, and the other two eigenvalues $\lambda_{2,3}$ are satisfy the equation

$$\lambda^2 - a_1\lambda + a_2 = 0.$$

By lemma (2.2), the following results can be obtained.

Theorem 2.2. *The non-trivial equilibrium point* $E_1(1, 1, 0)$ *is*

- *a stable if* $m_6 > 0, a_1 < 0, a_2 > 0$*;*
- *a hyperbolic saddle points, if the parameters satisfy one of the following conditions*

   1. $m_6 > 0, a_1 > 0, a_2 > 0$;
   2. $m_6 > 0, a_2 < 0$;

- *a non-hyperbolic point if the parameters satisfy one of the following conditions*
   1. $m_6 = 0, a_1 \neq 0, a_2 > 0$;
   2. $m_6 = 0, a_2 < 0$;
   3. $m_6 > 0, a_1 \neq 0, a_2 = 0$;
   4. $m_6 > 0, a_1 = 0, a_2 > 0$;
   5. $m_6 > 0, a_1 = a_2 = 0$;
   6. $m_6 = 0, a_1 \neq 0, a_2 = 0$;
   7. $m_6 = a_1 = 0, a_2 > 0$;
   8. $m_6 = a_1 = a_2 = 0$.

The characteristic equation (2.7) is written as

$$f(\lambda) = \lambda^3 + A_1\lambda^2 + B_1\lambda + C_1 = 0 \ , \tag{2.8}$$

where

$$A_1 = m_6m_7 - a_1, \ B_1 = a_2 - a_1m_6m_7, \ C_1 = a_2m_6m_7.$$

By the criterion of Routh-Hurwitz, the real part of the characteristic roots are all negative if and only if

$$\begin{cases} m_6m_7 - a_1 > 0, \\ (m_6m_7 - a_1)(a_2 - a_1m_6m_7) - (a_2m_6m_7) > 0, \\ a_2m_6m_7 > 0. \end{cases}$$

The above analysis can be summarized as the following theorem.

Theorem 2.3. *The equilibrium point E_1 of system (2.3) is asymptotically stable if and only if $A_1 > 0$, $A_1B_1 - C_1 > 0$, $C_1 > 0$.*

2.3. Bifurcations analysis of trivial equilibrium

In this part, we study the bifurcations of the system at the trivial equilibrium.

Theorem 2.4. *In case $m_1 = 1, m_6 > 0$, then system* (2.3) *undergoes Hopf bifurcation at the equilibrium point E_0.*

Proof. In order to analyze the bifurcation for system, we bring the $E_0(1, 1, 0)$ into the origin , let $\bar{x} = x - 1, \bar{y} = y - 1, \bar{z} = z$, the original system becomes

$$\begin{cases} \frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau} = m_1\big(-\bar{x} + (m_4 - 2)\bar{y} + m_2\bar{z} - 2\bar{x}\bar{y} + m_2\bar{x}\bar{z} - \bar{y}^2 + m_3\bar{z}^2 - \bar{x}\bar{y}^2 + m_3\bar{x}\bar{z}^2\big), \\ \frac{\mathrm{d}\bar{y}}{\mathrm{d}\tau} = \bar{x} + \bar{y} + m_5\bar{z} + 2\bar{x}\bar{y} + m_5\bar{x}\bar{z} + \bar{y}^2 + \bar{x}\bar{y}^2 - m_3\bar{z}^2 - m_3\bar{x}\bar{z}^2, \\ \frac{\mathrm{d}\bar{z}}{\mathrm{d}\tau} = m_6\bar{z}(m_7 - \bar{z}). \end{cases} \tag{2.9}$$

For the sake of convenience, we write it as the form

$$X = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}, \quad A = \begin{pmatrix} -m_1 & m_1(m_4-2) & m_1 m_2 \\ 1 & 1 & m_5 \\ 0 & 0 & m_6 m_7 \end{pmatrix},$$

$$F(X) = \begin{pmatrix} m_1(-2\bar{x}\bar{y} + m_2\bar{x}\bar{z} - \bar{y}^2 + m_3\bar{z}^2 - \bar{x}\bar{y}^2 + m_3\bar{x}\bar{z}^2) \\ 2\bar{x}\bar{y} + m_5\bar{x}\bar{z} + \bar{y}^2 + \bar{x}\bar{y}^2 - m_3\bar{z}^2 - m_3\bar{x}\bar{z}^2 \\ -m_6\bar{z}^2 \end{pmatrix}.$$

Therefore, system (2.9) can be written as

$$\dot{X} = AX + F(X). \tag{2.10}$$

Next, we prove that system undergoes the Hopf bifurcation at the origin if $m_1 = 1, m_6 > 0$. It is not difficult to find that the Jacobian matrix $J(E)$ of system (4.2) at the origin has three eigenvalues as the form

$$\lambda_{1,2} = \frac{1-m_1}{2} \pm \frac{\sqrt{(1-m_1)^2 - 4m_1(1-m_4)}}{2}, \qquad \lambda_3 = m_6 m_7.$$

In order to prove the appearance of the Hopf bifurcation, we need to verify the cross-sectional condition of the Hopf bifurcation in the system. Recall that $\emptyset = \frac{1-m_1}{2}$ is the real part of a complex root of $\lambda_{1,2}$. Let $m_1 = 1, m_6 > 0$, immediately, the matrix A has a pair of single complex eigenvalues on the imaginary axis,

$$\lambda_{1,\ 2} = \pm i\sqrt{1-m_4}.$$

Therefore, we select m_1 as the bifurcation parameter, which satisfies

$$\frac{d\emptyset}{dm_1} = -\frac{1}{2} \neq 0.$$

In addition to transversality condition, also, we need to verify that the first Lyapunov coefficient $l_1 \neq 0$. Let us recall that l_n is the nth Lyapunov coefficient of the equilibrium E_0. Next, we will give a method for calculating the first Lyapunov coefficient l_1 of the equilibrium point E_0. For convenience of calculation, we will analyze the topologically equivalent system (2.10) .

As can be seen from the theorem (3.1) in the references [3], the above system can be expressed as

$$\dot{X} = AX + \frac{1}{2}\bar{\mathcal{B}}(X,\ X) + \frac{1}{6}\bar{\mathcal{C}}(X,\ X,\ X),$$

since

$$A = \begin{pmatrix} -1 & (m_4-2) & m_2 \\ 1 & 1 & m_5 \\ 0 & 0 & m_6 m_7 \end{pmatrix},$$

multi-linear function $\bar{\mathcal{B}}$ and $\bar{\mathcal{C}}$ are Taylor expansion of function F(X) at origin.

To obtain the first Lyapunov coefficient, we should calculate the eigenvectors of matrix A for the eigenvalues $\lambda_{1,2}$ and the eigenvector of matrix A^T. Let the complex eigenvector q_{01} corresponds to complex eigenvalues λ_2, and p_{01} is the conjugate eigenvector, which satisfies $< p_{01}, q_{01} >= 1$.

According to $Aq_{01} = \lambda_1 q_{01}, A^T p_{01} = \lambda_2 p_{01}$, we get

$$q_{01} = c_1 \begin{pmatrix} 1 - i\sqrt{1-m_4} \\ -1 \\ 0 \end{pmatrix}, \quad p_{01} = c_2 \begin{pmatrix} -m_6 m_7 - i\sqrt{1-m_4} \\ (m_6 m_7 + i\sqrt{1-m_4})(-1 + i\sqrt{1-m_4}) \\ m_2 + m_5 - i m_5 \sqrt{1-m_4} \end{pmatrix},$$

where

$$c_1 = \sqrt{1-m_4} - i m_6 m_7,$$

$$c_2 = \frac{1}{2\sqrt{1-m_4}(1 - m_4 + m_6^2 m_7^2)}.$$

Then, we calculate

$$A^{-1}B(q_{01},\ \bar{q_{01}}) = -2(1 - m_4 + m_6^2 m_7^2) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$B(q_{01},\ A^{-1}B(q_{01},\ \bar{q_{01}})) = 4(1 - m_4 + m_6^2 m_7^2)(\sqrt{1-m_4} - i m_6 m_7) \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

$$(2i\emptyset_0 I_3 - A)^{-1}B(q_{01},\ q_{01}) = \frac{2}{3\sqrt{1-m_4}} c_1^2 (1 - 2i\sqrt{1-m_4}) \begin{pmatrix} -\sqrt{1-m_4} - 2i \\ 2i \\ 0 \end{pmatrix},$$

$$\begin{aligned} &B(\bar{q_{01}},\ (2i\emptyset_0 I_3 - A)^{-1}B(q_{01},\ q_{01})) \\ &= -\frac{4c_1^2}{3(1-m_4)}(i + 2\sqrt{1-m_4})(i + 2\sqrt{1-m_4} - i m_4)(\sqrt{1-m_4} + i m_6 m_7) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \end{aligned}$$

$$\bar{\mathcal{C}}(q_{01},\ q_{01},\ \bar{q_{01}}) = -2c_1(1 - m_4 + m_6^2 m_7^2)(3 - i\sqrt{1-m_4}) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

The expression of the first Lyapunov coefficient can be obtained as follows

$$\begin{aligned} l_1 =& \frac{1}{2\emptyset_0} Re[\langle p_{01},\ C(q_{01},\ q_{01},\ \bar{q_{01}})\rangle - 2\langle p_{01},\ B(q_{01},\ A^{-1}B(q_{01},\ \bar{q_{01}}))\rangle \\ &+ \langle p_{01},\ B(\bar{q_{01}},\ (2i\emptyset_0 I_3 - A)^{-1}B(q_{01},\ q_{01}))\rangle], \\ =& \frac{1}{2\sqrt{1-m_4}}(1 - m_4 - 2m_6m_7 + 2m_4m_6m_7 + 3m_6^2m_7^2), \\ =& \frac{1}{2\sqrt{1-m_4}}((1-m_4)(1-2m_6m_7) + 3m_6^2m_7^2). \end{aligned}$$

Based on Theorem 1 in [8], system (2.3) undergoes

- supercritical Hopf bifurcation if $1 - m_4 < \Gamma$,
- subcritical Hopf bifurcation if $1 - m_4 > \Gamma$,
- degenerate Hopf bifurcation if $1 - m_4 = \Gamma$.

where, $\Gamma = \frac{3m_6^2m_7^2}{2m_6m_7-1}$ □

Theorem 2.5. *In case $m_1 = 1, m_6 = 0$, then the system (2.3) undergoes degenerate fold-Hopf bifurcation at equilibrium E_0.*

Proof. Let $m_1 = 1 + \alpha_1, m_6 = \alpha_2$, then

$$A = \begin{pmatrix} -(1+\alpha_1) & (1+\alpha_1)(m_4-2) & (1+\alpha_1)m_2 \\ 1 & 1 & m_5 \\ 0 & 0 & \alpha_2 m_7 \end{pmatrix},$$

$$F(X) = \begin{pmatrix} (1+\alpha_1)(2\bar{x}\bar{y} + m_2\bar{x}\bar{z} - \bar{y}^2 + m_3\bar{z}^2 - \bar{x}\bar{y}^2 + m_3\bar{x}\bar{z}^2) \\ 2\bar{x}\bar{y} + m_5\bar{x}\bar{z} + \bar{y}^2 + \bar{x}\bar{y}^2 - m_3\bar{z}^2 - m_3\bar{x}\bar{z}^2 \\ -\alpha_2\bar{z}^2 \end{pmatrix}.$$

The characteristic equation of the system (4.2) at the origin is written as the following form

$$(\lambda - \alpha_2 m_7)[\lambda^2 + \alpha_1\lambda + (1+\alpha_1)(1-m_4)] = 0.$$

Thus, the equation has a root $\lambda_1 = \alpha_2 m_7$ and two conjugate roots $\lambda_{2,3} = \mu \pm i\nu$, where $\mu,\ \nu$ are define $\mu = -\frac{\alpha_1}{2},\ \nu = \sqrt{(\alpha_1+1)(1-m_4) - (\frac{\alpha_1}{2})^2}$.

For eigenvalues $\lambda_1,\ \lambda_2$, we can obtain the corresponding eigenvectors $q_{03},\ q_{13}$, and their conjugate eigenvectors $p_{03},\ p_{13}$, which satisfy

$$< p_{03}, q_{03} >=< p_{13}, q_{13} >= 1.$$

By computation

$$Aq_{03} = \lambda_1 q_{03},\ Aq_{13} = \lambda_2 q_{13},\ A^T p_{03} = \lambda_1 p_{03},\ A^T p_{13} = \lambda_3 p_{13},$$

we get

$$q_{03} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ -\rho_3 \end{pmatrix},\quad q_{13} = \begin{pmatrix} \lambda_2 - 1 \\ 1 \\ 0 \end{pmatrix},\quad p_{03} = -\frac{1}{\rho_3}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$p_{13} = \frac{1}{\lambda_3(2 - 2\alpha_2 m_7 + \alpha_1)} \begin{pmatrix} \lambda_3 - \alpha_2 m_7 \\ (1 + \alpha_1 + \lambda_3)(\lambda_3 - \alpha_2 m_7) \\ (1 + \alpha_1)(m_2 + m_5) + m_5 \lambda_3 \end{pmatrix},$$

where

$$\rho_1 = (1 + \alpha_1)((-2 + m_4)m_5 + m_2(-1 + m_7\alpha_2)),$$
$$\rho_2 = (1 + \alpha_1)(m_2 + m_5) + \alpha_2 m_5 m_7,$$
$$\rho_3 = (1 + \alpha_1)(m_4 - 1) - \alpha_2 m_7(\alpha_1 + \alpha_2 m_7).$$

Let $X = uq_{03} + vq_{13} + \bar{v}\bar{q_{13}}$, then, the system (2.10) turn into

$$\begin{cases} \dot{u} = \Gamma(\alpha) + \alpha_2 m_7 u + g(u, v, \bar{v}, \alpha), \\ \dot{v} = \Omega(\alpha) + (\mu(\alpha) + \nu(\alpha))v + h(u, v, \bar{v}, \alpha), \end{cases}$$

where

$$g(u, v, \bar{v}, \alpha) = \sum_{j+k+l\geq 2} \frac{1}{j!k!l!} g_{jkl}(\alpha) u^j v^k \bar{v}^l =< p_{03}(\alpha), F(uq_{03} + vq_{13} + \bar{v}\bar{q_{13}}, \alpha) >,$$

$$h(u, v, \bar{v}, \alpha) = \sum_{j+k+l\geq 2} \frac{1}{j!k!l!} h_{jkl}(\alpha) u^j v^k \bar{v}^l =< p_{13}(\alpha), F(uq_{03} + vq_{13} + \bar{v}\bar{q_{13}}, \alpha) > .$$

Due to the complexity of calculation of the real number $g_{jkl}(\alpha)$ and the complex number $h_{jkl}(\alpha)$, we will not give them specifically here. One can calculate $g_{200} = 0$, which means that the system (2.3) has a degenerate fold-Hopf bifurcation [21]. □

2.4. Bifurcation analysis of nontrivial equilibrium

In this part, we give a theoretical proof that the system undergoes fold bifurcation and Hopf bifurcation. For the sake of convenience, we write system (2.3) as the form

$$U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & -m_6 m_7 \end{pmatrix}, \quad F(U) = \begin{pmatrix} m_1(\Lambda_1 + \Lambda_2) \\ -\Lambda_1 + \Lambda_3 \\ -m_6 w^2 \end{pmatrix},$$

$$\Lambda_1 = -2y_* uv - \kappa(y_* - 1)v^2 + \kappa m_3(y_* - 1)w^2 - uv^2 + m_3 uw^2,$$
$$\Lambda_2 = (m_2 + 2m_3 m_7)uw, \ \Lambda_3 = (m_5 - 2m_3 m_7)uw$$

Therefore, the system is converted to

$$\dot{U} = AU + F(U). \tag{2.11}$$

Theorem 2.6. *When $a_2 = 0$, $m_6 a_1 \neq 0$, then system (2.3) undergoes fold bifurcation at the point E_1 if one of the following conditions is satisfied*

1. $a_1 + m_6 m_7 \neq 0$;
2. $a_1 + m_6 m_7 = 0$, $a_{11}[2a_{12}y_* - \kappa a_{11}(y_* - 1)] \neq 0$;

3. $a_1 + m_6 m_7 = 0,\ a_{11} = 0,\ 2a_{22}y_* - \kappa a_{21}(y_* - 1) \neq 0.$

Proof. $C_1 = 0$, i.e. $m_6 = 0$ or $a_2 = 0$, then the system (2.11) at the origin has the following form

$$\lambda(\lambda^2 + A_1\lambda + B_1) = 0. \tag{2.12}$$

Then, equation (2.11) has one zero root $\lambda_1 = 0$, and two roots λ_2, λ_3 are not equal to zero unless $(A_1,\ B_1)$ are equal to zero.

Let V_1, V_2 and V_3, are eigenvectors corresponding to $\lambda_1 = 0$, λ_2 and λ_3, where

$$V_1 = \begin{pmatrix} V_{11} \\ V_{12} \\ V_{13} \end{pmatrix},\ V_2 = \begin{pmatrix} V_{21} \\ V_{22} \\ V_{23} \end{pmatrix},\ V_3 = \begin{pmatrix} V_{31} \\ V_{32} \\ V_{33} \end{pmatrix},$$

hence

$$P = \begin{pmatrix} V_1 & V_2 & V_3 \end{pmatrix} = \begin{pmatrix} V_{11} & V_{21} & V_{31} \\ V_{12} & V_{22} & V_{32} \\ V_{13} & V_{23} & V_{33} \end{pmatrix},$$

$$P^{-1} = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix},\ U = P \begin{pmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{pmatrix},$$

and

$$\begin{cases} \dot{\bar{u}} = \sum_{i=1}^{3} P_{1i} F_i(\bar{u}, \bar{v}, \bar{w}), \\ \dot{\bar{v}} = \sum_{j=1}^{3} P_{2j} F_j(\bar{u}, \bar{v}, \bar{w}), \\ \dot{\bar{w}} = \sum_{k=1}^{3} P_{3k} F_k(\bar{u}, \bar{v}, \bar{w}), \end{cases} \tag{2.13}$$

where

$$\begin{cases} F_1(\bar{u}, \bar{v}, \bar{w}) = \sum_{2 \le j+k+l \le 3} f_{jkl} m_1 \bar{u}^j \bar{v}^k \bar{w}^l, \\ F_2(\bar{u}, \bar{v}, \bar{w}) = \sum_{2 \le j+k+l \le 3} \bar{f}_{jkl} \bar{u}^j \bar{v}^k \bar{w}^l, \\ F_3(\bar{u}, \bar{v}, \bar{w}) = m_6(V_{13}\bar{u} + V_{23}\bar{v} + V_{33}\bar{w})^2, \end{cases}$$

$$\begin{cases} f_{200} = -2y_* V_{11} V_{12} + (m_2 + 2m_3 m_7) V_{11} V_{13} - \kappa(y_* - 1) V_{12}^2 + \kappa m_3 (y_* - 1) V_{13}^2, \\ \bar{f}_{200} = 2y_* V_{11} V_{12} + (m_5 - 2m_3 m_7) V_{11} V_{13} + \kappa(y_* - 1) V_{12}^2 - \kappa m_3 (y_* - 1) V_{13}^2. \end{cases}$$

It is easy to get, the coefficient of $\bar{u}^2$

$$\hbar = P_{11} m_1 f_{200} + P_{12} \bar{f}_{200} + P_{13} m_6 V_{13}^2.$$

Applying bifurcation Theory [11], the system (2.3) undergoes fold bifurcation if $\hbar \neq 0$.

First, we prove that $m_6 = 0, a_1 a_2 \neq 0$, then we will discuss it in two categories.

First case, we consider that $\lambda_2 \neq \lambda_3$, i.e. $A_1^2 - 4B_1 \neq 0$, it's easy to get

$$V_1 = \begin{pmatrix} a_4 \\ -a_3 \\ a_2 \end{pmatrix},\ V_2 = \begin{pmatrix} a_{12} \\ \lambda_2 - a_{11} \\ 0 \end{pmatrix},\ V_3 = \begin{pmatrix} \lambda_3 - a_{22} \\ a_{21} \\ 0 \end{pmatrix},$$

where

$$a_3 = a_{11}a_{23} - a_{13}a_{21}, a_4 = a_{12}a_{23} - a_{13}a_{22},$$

$$P = \begin{pmatrix} a_4 & a_{12} & \lambda_3 - a_{22} \\ -a_3 & \lambda_2 - a_{11} & a_{21} \\ a_2 & 0 & 0 \end{pmatrix},$$

$$P^{-1} = \frac{1}{\Delta}\begin{pmatrix} 0 & 0 & \frac{1}{a_2}\Delta \\ -a_{21} & \lambda_3 - a_{11} & \frac{1}{a_2}(a_{21}a_4 + a_3\lambda_3 - a_{11}a_3) \\ \lambda_2 - a_{11} & -a_{12} & \frac{1}{a_2}(a_{11}a_4 - a_4\lambda_2 - a_{12}a_3) \end{pmatrix},$$

where $\Delta = Det(P) = a_{11}^2 - a_{12}a_{21} + \lambda_2\lambda_3 - a_{11}(\lambda_2 + \lambda_3)$.
By calculating

$$f_{200} = a_4(a_2(m_2 + 2m_3m_7) + 2a_3y_*) + (-a_3^2 + a_2^2m_3)(-1 + y_*)\kappa,$$

$$\bar{f}_{200} = a_4(a_2(m_5 - 2m_3m_7) - 2a_3y_*) + (a_3^2 - a_2^2m_3)(-1 + y_*)\kappa,$$

$$F_3(\bar{u}, \bar{v}, \bar{w}) = 0.$$

Here, $\bar{u}^2$ has zero coefficients.

Second case, we consider that $\lambda_2 = \lambda_3$, i.e. $a_{12}a_{21} = 0, a_{11} = a_{22} \neq 0$, $\lambda_2 = \frac{a_1}{2} = \sqrt{a_2}$, as in the previous method, we obtain that $\bar{u}^2$ has zero coefficients.

Finally, when $a_2 = 0, m_6 \neq 0, a_1 \neq 0$ is proved, then there is $\lambda_2 = -m_6m_7, \lambda_3 = a_1$, and then, we will discuss it in two categories. Firstly, we consider that $\lambda_2 \neq \lambda_3$, i.e. $a_1 + m_6m_7 \neq 0$, therefore

$$V_1 = \begin{pmatrix} a_{12} \\ -a_{11} \\ 0 \end{pmatrix}, V_2 = \begin{pmatrix} -m_6m_7a_{13} + a_4 \\ -m_6m_7a_{23} - a_3 \\ m_6m_7(m_6m_7 + a_1) \end{pmatrix}, V_3 = \begin{pmatrix} a_{11} \\ a_{21} \\ 0 \end{pmatrix},$$

where

$$a_3 = a_{11}a_{23} - a_{13}a_{21}, a_4 = a_{12}a_{23} - a_{13}a_{22},$$

$$P = \begin{pmatrix} a_{12} & m_6m_7a_{13} + a_4 & a_{11} \\ -a_{11} & m_6m_7a_{23} - a_3 & a_{21} \\ 0 & m_6m_7(m_6m_7 + a_1) & 0 \end{pmatrix},$$

$$P^{-1} = \frac{1}{a_{11}a_1}\begin{pmatrix} -a_{21} & a_{11} & \frac{a_3}{m_6m_7} \\ 0 & 0 & -\frac{a_{11}a_1}{m_6m_7(m_6m_7+a_1)} \\ -a_{11} & -a_{12} & -\frac{a_5}{m_6m_7+a_1} \end{pmatrix},$$

where

$$a_5 = a_{12}a_{23} + a_{11}a_{13}.$$

Through calculation, we get

$$f_{200} = 2y_*a_{11}a_{12} - \kappa(y_* - 1)a_{11}^2,$$

$$\bar{f}_{200} = -2y_* a_{11} a_{12} + \kappa(y_* - 1) a_{11}^2,$$

$$\begin{aligned}\hbar &= P_{11} m_1 f_{200} + P_{12} \bar{f}_{200} + P_{13} m_6 V_{13}^2 \\ &= \frac{a_{11}}{a_1}[-2y_*(m_1 a_{22} + a_{12}) + \kappa(y_* - 1)(m_1 a_{21} + a_{11})] \\ &= \frac{a_{11} m_1}{a_1}[2y_*(1 - m_4) + \kappa m_7 (y_* - 1)(m_2 + m_5)].\end{aligned}$$

Due to $y_* > 1, \kappa > 0, m_7 > 0, m_2 + m_5 > 0, 0 \leq m_4 < 0$, then $\hbar \neq 0$.

It can be proved that when $a_2 = 0, m_6 \neq 0, a_1 \neq 0$, and $a_1 + m_6 m_7 \neq 0$, system undergoes fold bifurcation.

Secondly, we consider that $\lambda_2 = \lambda_3$, i.e. $a_1 + m_6 m_7 = 0, a_{23} a_{22} + a_{13} a_{21} = 0$, As with the above method, when $a_{11} \neq 0$, we calculate

$$V_1 = \begin{pmatrix} a_{12} \\ -a_{11} \\ 0 \end{pmatrix}, \ V_2 = \begin{pmatrix} a_{11} \\ a_{21} \\ 0 \end{pmatrix}, \ V_3 = \begin{pmatrix} 0 \\ a_{23} \\ a_{11} \end{pmatrix},$$

where

$$P = \begin{pmatrix} a_{12} & a_{11} & 0 \\ -a_{11} & a_{21} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}, \quad P^{-1} = \frac{1}{a_{11} a_1} \begin{pmatrix} a_{21} & -a_{11} & a_{23} \\ a_{11} & a_{12} & -\frac{a_{12} a_{23}}{a_{11}} \\ 0 & 0 & a_1 \end{pmatrix}.$$

Through calculations, we get

$$f_{200} = 2y_* a_{11} a_{12} - \kappa(y_* - 1) a_{11}^2,$$

$$\bar{f}_{200} = -2y_* a_{11} a_{12} + \kappa(y_* - 1) a_{11}^2,$$

$$\hbar = -\frac{m_1}{m_6}(m_2 + m_5)(2a_{12} y_* - \kappa a_{11}(y_* - 1)),$$

which proves that, if $2a_{12} y_* - \kappa a_{11}(y_* - 1) \neq 0$, then it undergoes fold bifurcation.

When $a_{11} = 0$, we get

$$P = \begin{pmatrix} a_{22} & 0 & a_{13} \\ -a_{21} & 1 & 0 \\ 0 & 0 & a_{22} \end{pmatrix}, \quad P^{-1} = \frac{1}{a_{22}} \begin{pmatrix} a_{22} & 0 & -\frac{a_{13}}{a_{22}} \\ a_{21} & a_{22} & -\frac{a_{13} a_{21}}{a_{22}} \\ 0 & 0 & a_{22} \end{pmatrix},$$

$$\hbar = m_1 a_{21}(2a_{22} y_* - \kappa a_{21}(y_* - 1)).$$

If $\hbar \neq 0$, then system (2.3) undergoes fold bifurcation at E_1.

In summary, when $a_2 = 0, m_6 a_1 \neq 0$ then system (2.3) bifurcates from the point E_1, if one of the conditions of theorem (2.6) is satisfied. □

Theorem 2.7. *If $a_1 = 0$, $a_2 > 0$, the system* (2.3) *undergoes Hopf bifurcation at the non-trivial equilibrium E_1.*

Proof. From formula (2.7), one can find easily that the Jacobian matrix $J(E)$ of system (2.11) has three eigenvalues as the following forms

$$\lambda_1 = -m_6 m_7, \qquad \lambda_{2,3} = \frac{a_1}{2} \pm \frac{\sqrt{a_1^2 - 4a_2}}{2}.$$

In order to prove the appearance of the Hopf bifurcation, we need to verify the cross-sectional condition of the Hopf bifurcation in the system. Recall that $\chi = \frac{a_1}{2}$ be the real part of a complex root of the characteristic equation (3.5),If $a_1(y_*) = 0$ and $a_2(y_*) > 0$. In order to preserve the generality, we select $m_{30} = m_{30}(m_3)$ as the bifurcation parameter, which satisfies

$$a_1(y_*(m_{30}),\ m_{30}) = 0, \qquad a_2(y_*(m_{30}),\ m_{30}) > 0.$$

At this point, the system (5.1) has two eigenvalues $\lambda_{2,\ 3} = \pm i\sqrt{a_2(y_*(m_{30}),\ m_{30})}$ where i represents an imaginary unit.

Due to

$$\frac{d\chi}{dm_3} = \frac{1}{2} a_1^{'}(y_*(m_{30}),\ m_{30}) = \frac{1}{2} m_1 m_7^2 \neq 0,$$

the system creates a cross-sectional condition for the Hopf bifurcation at the equilibrium.

Next, we calculate the first Lyapunov coefficient $l_1$of the equilibrium point E_1. For convenience of calculation, we will analyze the topologically equivalent system (5.1).

By the straightforward calculation, the characteristic equation at the origin with $a_1 = 0, m_6 > 0, a_2 > 0$, can be given by

$$(\lambda + m_6 m_7)(\lambda^2 + a_2) = 0,$$

Assume that, the characteristic roots are respectively

$$\lambda_{1,2} = \pm i\varrho(\varrho = \sqrt{a_2}), \lambda_3 = -m_6 m_7.$$

We can see that the above system can be expressed as

$$\dot{U} = AU + \frac{1}{2}\mathfrak{B}(U,\ U) + \frac{1}{6}\mathfrak{C}(U,\ U,\ U),$$

where, $A = A(m_{30})$ and multi-linear function $\mathcal{B}$ and $\mathcal{C}$ can be written as

$$\mathfrak{B}(\bar{\xi},\ \bar{\eta}) = \begin{pmatrix} m_1[-2y_*\bar{\rho_1} + (m_2 + 2m_3 m_7)\bar{\rho_2} - 2\kappa(y_* - 1)\bar{\rho_3} + 2\kappa m_3(y_* - 1)\bar{\rho_4}] \\ 2y_*\bar{\rho_1} + (m_5 - 2m_3 m_7)\bar{\rho_2} + 2\kappa(y_* - 1)\bar{\rho_3} - 2\kappa m_3(y_* - 1)\bar{\rho_4} \\ -2m_6\bar{\rho_4} \end{pmatrix},$$

$$\mathfrak{C}(\bar{\xi},\ \bar{\eta},\ \bar{\zeta}) = \begin{pmatrix} 2m_1(-\bar{\rho_5} + m_3\bar{\rho_6}) \\ 2(\bar{\rho_5} - m_3\bar{\rho_6}) \\ 0 \end{pmatrix}.$$

For planar vector $\bar{\xi} = (\bar{\xi_1},\ \bar{\xi_2},\ \bar{\xi_3})^T$, $\bar{\eta} = (\bar{\eta_1},\ \bar{\eta_2}, \bar{\eta_3})^T$ and $\bar{\zeta} = (\ \bar{\zeta_1},\ \bar{\zeta_2}, \bar{\zeta_3})^T$. we have

$$\bar{\rho_1} = \bar{\xi_1}\bar{\eta_2} + \bar{\xi_2}\bar{\eta_1}, \qquad \bar{\rho_2} = \bar{\xi_1}\bar{\eta_3} + \bar{\xi_3}\bar{\eta_1}, \qquad \bar{\rho_3} = \bar{\xi_2}\bar{\eta_2},$$

$$\bar{\rho_4} = \bar{\xi_3}\bar{\eta_3}, \qquad \bar{\rho_5} = \bar{\xi_1}\bar{\eta_2}\bar{\zeta_2} + \bar{\xi_2}\bar{\eta_1}\bar{\zeta_2} + \bar{\xi_2}\bar{\eta_2}\bar{\zeta_1}, \qquad \bar{\rho_6} = \bar{\xi_1}\bar{\eta_3}\bar{\zeta_3} + \bar{\xi_3}\bar{\eta_1}\bar{\zeta_3} + \bar{\xi_3}\bar{\eta_3}\bar{\zeta_1}.$$

In the following, to obtain the first Lyapunov coefficient we should calculate the eigenvectors of matrix $A(m_{30})$ for the eigenvalues $\lambda_{2,3}$ and the eigenvector of matrix $A^T(m_{30})$. Let the complex eigenvector q_{11} corresponds to complex eigenvalues λ_2, and p_{11} is the conjugate eigenvector, which satisfies $< p_{11}, q_{11} >= 1$. $Aq_{11} = \lambda_1 q_{11}, A^T p_{11} = \lambda_2 p_{11}$, then we get

$$q_{11} = \bar{\phi_1}\begin{pmatrix} v - a_{12} \\ a_{11} - i\varrho \\ 0 \end{pmatrix}, \qquad p_{11} = \bar{\phi_2}\begin{pmatrix} -a_{21}(m_6 m_7 - i\varrho) \\ (m_6 m_7 - i\varrho)(a_{11} + i\varrho) \\ a_3 + ia_{23}\varrho \end{pmatrix},$$

where

$$a_3 = a_{11}a_{23} - a_{21}a_{13},$$

$$\bar{\phi_1} = (m_6 m_7 - i\varrho)(a_2 - ia_{11}\varrho),$$

$$\bar{\phi_2} = -\frac{1}{2a_2(m_6^2 m_7^2 + a_2)(a_2 + a_{11}^2)}.$$

The expression of the first Lyapunov coefficient can be obtained as follows:

$$l_1 = \frac{1}{2\varrho} Re[\langle p_{11},\ \mathfrak{C}(q_{11},\ q_{11},\ \bar{q_{11}})\rangle - 2\langle p_{11},\ \mathfrak{B}(q_{11},\ A^{-1}\mathfrak{B}(q_{11},\ \bar{q_{11}}))\rangle$$
$$+ \langle p_{11},\ \mathfrak{B}(\bar{q_{11}},\ (2i\varrho I_3 - A)^{-1}\mathfrak{B}(q_{11},\ q_{11}))\rangle].$$

Overall, we can draw the following conclusions
(1) if $l_1 < 0$, the direction of bifurcation is supercritical;
(2) if $l_1 > 0$, the direction of bifurcation is subcritical;
(3) if $l_1 = 0$, system undergoes degenerate Hopf bifurcation. □

2.5. Numerical simulations

In this section, we use the numerical simulation method to calculate the first Lyapunov coefficient of the Hopf bifurcation. Selection of parameters as shown in Table 2, and the initial point is (0.85, 0.95, 0.18), the relevant simulations are shown in Figure 2(a)-(d).

Table 2. Coefficient of the system.

Parameter	m_1	m_2	m_3	m_4	m_5	m_6	m_7
Value	0.9	0.05	0.02	0.55	0.5	0.1	0.2

Table 3. The Calculation Results.

Fig.2	Hopf bifurcation point	The first Lyapunov coefficient
(a)	(0.784,1.192,0.2)	-0.311
(b)	(0.881,1.215,0.2)	-0.237
(c)	(0.944,1.048,0.2)	-0.239
(d)	(0.948, 1.039, 0.033)	-0.24

With m_1 as a free parameter, the bifurcation diagram is shown in Figure 2(a). As we can be seen from the graph, when the parameter m_1 changes to 0.617, Hopf bifurcation occurs in the system (2.3), and the first Lyapunov coefficient $l_1 = -0.311 < 0$. This means that the system will generate a supercritical Hopf bifurcation under this set of parameters. Recall this equilibrium point is $H_1(0.784, 1.192, 0.2)$, which is the supercritical Hopf bifurcation point.

There are respectively m_3, m_5, m_7 is a bifurcation graph of free parameters in Figure 2(b)-(d). It can be seen from the figure that there is a Hopf bifurcation point in the system (2.3), and the value of the first Lyapunov coefficient can be obtained by calculation. The results are shown in Table 3 below, and the system undergoes a supercritical Hopf bifurcation.

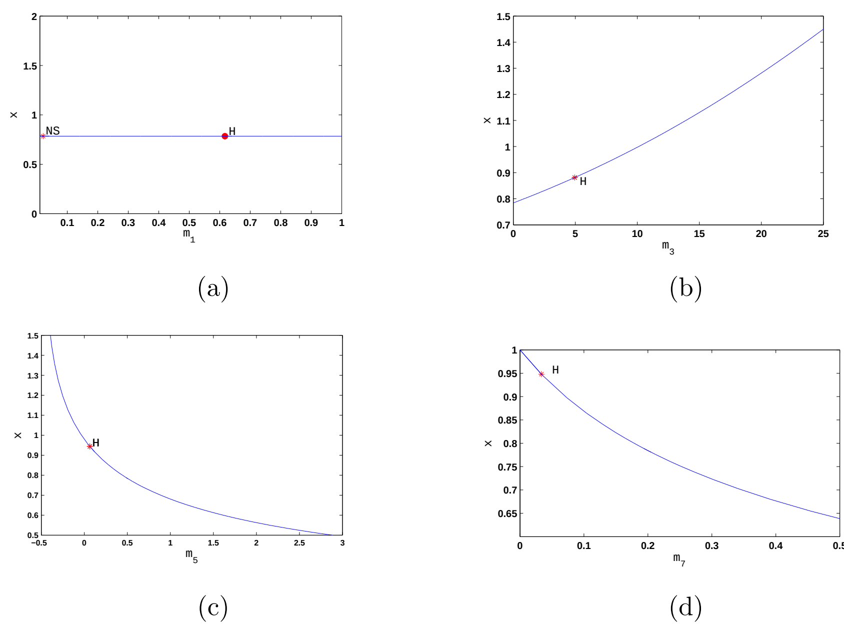

Figure 2. Bifurcation diagrams with m_1, m_3, m_5, m_7 as a free parameter respectively.

Table 4. Coefficient of the system.

Parameter	m_1	m_2	m_3	m_4	m_5	m_6	m_7
Value	1	0.5	0.02	0.55	0.05	0.001	1.5

Let's select another set of parameter values as shown in Table 4. With help of MATLAB, the time series diagrams of x, y and z varying with time are shown in Figure 3(a)-(c), and the phase diagram of system variables x and y varying with time is shown in Figure 3(d). At this time, the supercritical Hopf bifurcation of the system produces a stable limit cycle, which indicates when the advertising effect increases with time, the number of potential consumers and consumers changes periodically over a certain period of time.

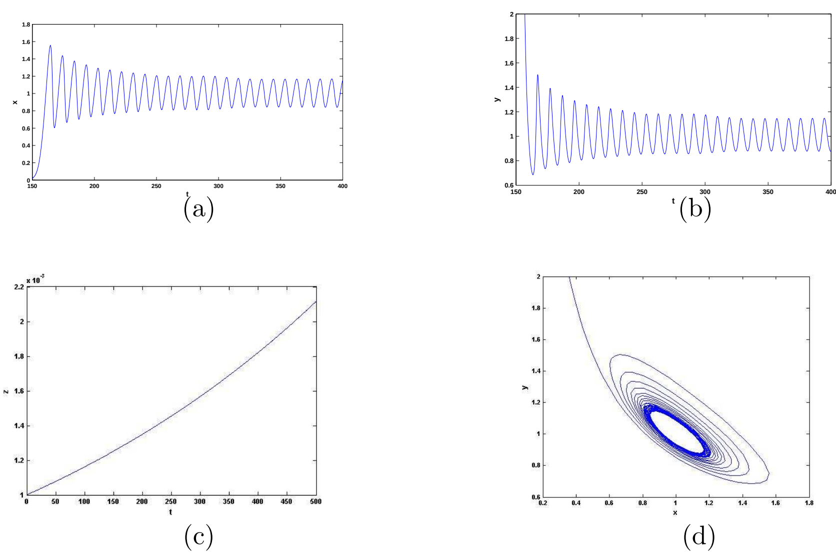

Figure 3. (a)-(c) Time series diagrams; (d) The phase diagram.

3. Dynamic analysis of the discrete consumption behavior model

However, in daily consumption, the consumer group variables and market advertising effects of washing products, food and other necessities change continuously with time; For luxury goods, cars and other general needs, the number of consumer groups and market advertising effects change periodically over time. Then the continuous consumption behavior model is discretized to obtain the following model

$$\begin{cases} x_1(n+1) = x_1(n) + k + bx_1(n)x_3(n) - ax_1(n)x_2^2(n) \\ \qquad\qquad - cx_1(n)x_3(n)(1-x_3(n)) + \beta x_2(n), \\ x_2(n+1) = x_2(n) + ax_1(n)x_2^2(n) + cx_1(n)x_3(n)(1-x_3(n)) - (\beta+\epsilon)x_2(n), \\ x_3(n+1) = x_3(n) + dx_3(n)(1-\frac{x_3(n)}{e}), \end{cases} \tag{3.1}$$

the parameters are the same as those described in the continuous model.

In order to facilitate the analysis of its dynamic behavior, we obtain its topological equivalent system through a series of transformations. Let $s_0 = (\beta+\epsilon),\ m =$

$(b-c)$, it's transformed into

$$\begin{cases} x_1(n+1) = x_1(n) + k + mx_1(n)x_3(n) - ax_1(n)x_2^2(n) + cx_1(n)x_3^2(n) + \beta x_2(n), \\ x_2(n+1) = x_2(n) + ax_1x_2^2 + cx_1(n)x_3(n)(1-x_3(n)) - s_0x_2(n), \\ x_3(n+1) = x_3(n) + dx_3(n)(1-\frac{x_3(n)}{e}), \end{cases} \tag{3.2}$$

here, the parameters a, b, k, ϵ, e are positive, β, c, d are non-negative and $s_0 \geq \epsilon$.

The variables of system (3.2) are linearly scaled as follows

$$x(n) = \frac{ak}{s_0\epsilon}x_1(n),\ y(n) = \frac{\epsilon}{k}x_2(n),\ z(n) = \frac{\epsilon}{k}x_3(n),$$

we get

$$\begin{cases} x(n+1) = x(n) + \frac{ak^2}{\epsilon^2}[1-x(n)y^2(n) + \frac{m\epsilon}{ak}x(n)z(n) + \frac{c}{a}x(n)z^2(n) + \frac{\beta}{s_0}(y(n)-1)], \\ y(n+1) = y(n) + s_0[x(n)y^2(n) - y(n) - \frac{c}{a}x(n)z^2 + \frac{c\epsilon}{ak}x(n)z(n)], \\ z(n+1) = z(n) + \frac{dk}{e\epsilon}z(n)(\frac{e\epsilon}{k} - z(n)), \end{cases} \tag{3.3}$$

Let $s_1 = \frac{ak^2}{\epsilon^2},\ s_2 = \frac{m\epsilon}{ak},\ s_3 = \frac{c}{a},\ s_4 = \frac{\beta}{s_0},\ s_5 = \frac{c\epsilon}{ak},\ s_6 = \frac{kd}{e\epsilon},\ s_7 = \frac{e\epsilon}{k}$, it is simplified to

$$\begin{cases} x(n+1) = x(n) + s_1[1 - x(n)y^2(n) + s_2x(n)z(n) + s_3x(n)z^2(n) + s_4(y(n)-1)], \\ y(n+1) = y(n) + s_0[x(n)y^2(n) - y(n) - s_3x(n)z^2(n) + s_5x(n)z(n)], \\ z(n+1) = z(n) + s_6z(n)(s_7 - z(n)), \end{cases} \tag{3.4}$$

where, $s_i > 0\ (i = 1,\ 7),\ s_j \geq 0\ (j = 3,\ 5,\ 6),\ 0 \leq s_4 < 1$, and $s_2 + s_5 > 0$.

Through linear transformation, system (3.1) is equivalent to system (3.4). The following analysis discusses the system (3.4) which is topologically equivalent.

3.1. Stability for fixed points

Similar to the method of system (2.4) analysis of equilibrium points, it is not difficult to analyse that system (3.4) exists a trivial fixed point $E^\circ(1,\ 1, 0)$, a positive non-trivial fixed point $E^*(x*,\ y*,\ Z*)$, which satisfies the following equation

$$\begin{cases} x^* = \frac{1-s_4}{s_7(s_2+s_5)}(y^*-1), \\ (y^*)^3 - (y^*)^2 + r^*y^* + h^* = 0, \\ z^* = s_7, \end{cases} \tag{3.5}$$

here

$$\begin{cases} r^* = \frac{s_7}{1-s_4}[s_3s_7(s_4-1) - s_2 - s_4s_5] = -\frac{e\epsilon(ms_0+c(\beta+e\epsilon))}{ak^2}, \\ h^* = s_7(s_3s_7 - s_5) = \frac{c(e-1)e\epsilon^2}{ak^2}, \\ r^* + h^* = -\frac{s_7}{1-s_4}(s_2+s_5) = -\frac{e\epsilon(ms_0+c(\beta+\epsilon))}{ak^2} < 0. \end{cases}$$

Next, we discuss stability of these two fixed points separately.

The Jacobian matrix of system (3.4) at any point $E(x, y, z)$ can be expressed as

$$J(E) = \begin{pmatrix} s_1(-y^2 + s_2z + s_3z^2) + 1 & s_1(-2xy + s_4) & s_1(s_2x + 2s_3xz) \\ s_0(y^2 - s_3z^2 + s_5z) & s_0(2xy-1) + 1 & s_0(-2s_3xz + s_5x) \\ 0 & 0 & s_6(s_7 - 2z) + 1 \end{pmatrix}$$

Therefore, the Jacobian matrix at the trivial fixed point E° can be expressed as

$$J(E^\circ) = \begin{pmatrix} -s_1+1 & s_1(-2+s_4) & s_1 s_2 \\ s_0 & s_0+1 & s_0 s_5 \\ 0 & 0 & s_6 s_7+1 \end{pmatrix}$$

That is, the characteristic equation of the fixed point E°

$$g(\lambda) = (\lambda - Q_0)(\lambda^2 + R_0\lambda + S_0) = 0, \tag{3.6}$$

where $Q_0 = s_6 s_7 + 1,\ R_0 = s_1 - s_0 - 2,\ S_0 = s_0 - s_1 + s_0 s_1(1 - s_4) + 1.$

By calculation, the eigenvalue at E° is $\lambda_1 = Q_0$, λ_i $(i = 2, 3)$ satisfy the equation

$$\lambda^2 + R_0\lambda + S_0 = 0.$$

In order to study stability of its fixed point, the following lemma is introduced.

Lemma 3.1. *[12] Let $F(\lambda) = \lambda^2 + B\lambda + C$, the two roots of λ_1, λ_2 are $F(\lambda) = 0$.*

1. *$|\lambda_1| < 1,\ |\lambda_2| < 1$ if $F(-1) > 0,\ C < 1$;*
2. *$|\lambda_1| < 1,\ |\lambda_2| > 1$ or $(|\lambda_1| > 1,\ |\lambda_2| < 1)$ if $F(-1) < 0$;*
3. *$|\lambda_1| > 1,\ |\lambda_2| > 1$ if $F(-1) > 0,\ C > 1$;*
4. *$\lambda_1 = 1,\ |\lambda_2| \neq 1$ if $F(1) = 0,\ B \neq 0, -2$;*
5. *$\lambda_1 = -1,\ |\lambda_2| \neq 1$ if $F(-1) = 0,\ B \neq 0, 2$;*
6. *$\lambda_1,\ \lambda_2$ is plural and $|\lambda_1| = |\lambda_2| = 1$ if $B^2 - 4C < 0,\ C = 1$.*

Note $G(\lambda) = \lambda^2 + R_0\lambda + S_0$,

$$\Phi_0 = G(-1) = 2(s_0 - s_1 + 2) + s_0 s_1(1 - s_4),$$

$$\Phi_1 = S_0 - 1 = s_0 - s_1 + s_0 s_1(1 - s_4).$$

By lemma(3.1), the following results can be obtained.

Theorem 3.1. *The trivial fixed point E° is*

- *a source if $s_6 > 0,\ \Phi_i > 0\ (i = 0,\ 1)$;*
- *a saddle if $s_6 > 0,\ \Phi_0 < 0$ or $s_6 > 0,\ \Phi_0 > 0,\ \Phi_1 < 0$;*
- *a non-hyperbolic if the parameter satisfies one of the following conditions*
  1. *$s_6 = 0,\ \Phi_0\Phi_1 \neq 0$;*
  2. *$s_6 = 0,\ \Phi_0 < 0,\ \Phi_1 = 0$;*
  3. *$s_6 > 0,\ \Phi_0 = 0,\ R_0 \neq 0,\ 2$;*
  4. *$s_6 > 0,\ \Phi_1 = 0,\ 0 < s_1 - s_0 < 4$;*
  5. *$s_6 = 0,\ \Phi_0 = 0,\ R_0 \neq 0,\ 2$;*
  6. *$s_6 = 0,\ \Phi_1 = 0,\ 0 < s_1 - s_0 < 4$.*

In order to facilitate the study of stability of the non-trivial fixed point E^*, we move the fixed point of the system to the origin

$$u = x(n) - x^*,\ v = y(n) - y^*,\ w = z(n) - z^*,$$

system (3.4) becomes

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} u + T_{11}u + T_{12}v + T_{13}w + \Xi_1 \\ v + T_{21}u + T_{22}v + T_{23}w + \Xi_2 \\ w - s_6 w(w + s_7) \end{pmatrix}, \tag{3.7}$$

here

$$\bar{\kappa} = \frac{1 - s_4}{s_7(s_2 + s_5)},$$

$$T_{11} = s_1[-(y^*)^2 + s_2 s_7 + s_3 s_7^2], \qquad T_{12} = s_1[-2\bar{\kappa}(y^*)^2 + 2\bar{\kappa} y^* + s_4],$$
$$T_{13} = s_1\bar{\kappa}(s_2 + 2s_3 s_7)(y^* - 1), \qquad T_{21} = s_0[(y^*)^2 + s_5 s_7 - s_3 s_7^2],$$
$$T_{22} = s_0[2\bar{\kappa} y^*(y^* - 1) - 1], \qquad T_{23} = s_0\bar{\kappa}(s_5 - 2s_3 s_7)(y^* - 1),$$

$$\begin{aligned}\Xi_1 =& s_1(-2y^* uv + (s_2 + 2s_3 s_7)uw - \bar{\kappa}(y^* - 1)v^2 \\ &+ \bar{\kappa} s_3(y^* - 1)w^2 - uv^2 + s_3 uw^2)\end{aligned}$$

$$\begin{aligned}\Xi_2 =& s_0(2y^* uv + (s_5 - 2s_3 s_7)uw + \bar{\kappa}(y^* - 1)v^2 \\ &- \bar{\kappa} s_3(y^* - 1)w^2 + uv^2 - s_3 uw^2)\end{aligned}$$

That is, the feature matrix at fixed point E^* is

$$J(E^*) = \begin{pmatrix} 1 + T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} + 1 & T_{23} \\ 0 & 0 & 1 - s_6 s_7 \end{pmatrix},$$

the characteristic equation is

$$h(\lambda) = (\lambda + s_6 s_7 - 1)(\lambda^2 - T_1\lambda + T_2) = 0,$$

and

$$T_1 = (2\bar{\kappa} s_0 - s_1)(y^*)^2 - 2s_0\bar{\kappa} y^* + s_1 s_7(s_2 + s_3 s_7) - s_0 + 2,$$
$$\begin{aligned}T_2 =& [3s_0 s_1(1 - s_4) + 2\bar{\kappa} s_0 - s_1]{y^*}^2 - 2[s_0 s_1(1 - s_4) + \bar{\kappa} s_0]y^* \\ &+ s_1 s_7[s_2 + s_3 s_7 - s_0(s_2 + s_4 s_5) + s_0 s_3 s_7(s_4 - 1)] + 1 - s_0.\end{aligned}$$

By calculation, the feature value at the fixed point E^* $\lambda_1 = 1 - s_6 s_7$, λ_i $(i = 2, 3)$ satisfies the equation

$$\lambda^2 - T_1\lambda + T_2 = 0.$$

Let

$$\begin{aligned}\Theta_0 =& [3s_0 s_1(1 - s_4) + 2(2\bar{\kappa} s_0 - s_1)]{y^*}^2 - 2s_0[s_1(1 - s_4) + 2\bar{\kappa}]y^* \\ &+ s_1 s_7[2(s_2 + s_3 s_7) - s_0(s_2 + s_4 s_5) + s_0 s_3 s_7(s_4 - 1)] + 4 - 2s_0,\end{aligned}$$

$$\Theta_1 =[3s_0s_1(1-s_4)+2\bar{\kappa}s_0-s_1]{y^*}^2-2[s_0s_1(1-s_4)+\bar{\kappa}s_0]y^* \\ +s_1s_7[s_2+s_3s_7-s_0(s_2+s_4s_5)+s_0s_3s_7(s_4-1)]-s_0,$$

$$\Theta_2 =3s_0s_1(1-s_4){y^*}^2-2s_0s_1(1-s_4)y^* \\ +s_0s_1s_7[-(s_2+s_4s_5)+s_3s_7(s_4-1)].$$

By lemma (3.1), we get the following result

Theorem 3.2. *The non-trivial fixed point E^* is*

- *a sink if* $0 < s_6s_7 < 2,\ \Theta_0 > 0,\ \Theta_1 < 0$;
- *a source if* $s_6s_7 > 2,\ \Theta_i > 0\ (i = 0,\ 1)$;
- *a saddle if one of the following conditions holds*
  1. $0 < s_6s_7 < 2,\ \Theta_i > 0\ (i = 0,\ 1)$;
  2. $s_6s_7 > 2,\ \Theta_0 > 0,\ \Theta_1 < 0$;
  3. $s_6s_7 \neq 0,\ 2,\ \Theta_0 < 0$;
- *a non-hyperbolic if one of the following conditions holds*
  1. $s_6s_7 = 0, \Theta_0\Theta_1 \neq 0$ *or* $\Theta_0 < 0,\ \Theta_1 = 0\hat{A}£\hat{A}$
  2. $s_6s_7 \neq 0,\ 2,\ \Theta_2 = 0,\ T_1 \neq 0,\ 2;$
  3. $s_6s_7 = 2, \Theta_0\Theta_1 \neq 0$ *or* $\Theta_0 < 0,\ \Theta_1 = 0$;
  4. $s_6s_7 \neq 0,\ 2,\ \Theta_1 = 0,\ T_1 \neq 0,\ -2;$
  5. $s_6s_7 \neq 0,\ 2,\ T_1 = 0,\ T_2 = -1;$
  6. $s_6s_7 \neq 0,\ 2,\ -2 < T_1 < 2,\ T_2 = 1;$
  7. $s_6s_7 = 0,\ -2 < T_1 < 2,\ T_2 = 1;$
  8. $s_6s_7 = 2,\ -2 < T_1 < 2,\ T_2 = 1;$
  9. $s_6s_7 = 0,\ T_1 = 0,\ T_2 = -1;$
  10. $s_6s_7 = 0,\ T_1 = 0,\ T_2 = -1.$

3.2. Flip bifurcation

In this part, we prove existence of flip bifurcation and get the theorem (3.3).

Theorem 3.3. *If parameters satisfy* $FL = \{s_i\ (i = 1,\ 2, ..., 7)|s_6 > 0,\ \Phi_0 = 0,\ R_0 \neq 0,\ 2\}$, *system (3.4) undergoes flip bifurcation. In addition, note* $\Phi_2 = s_1 - s_0 - 4$, *if* $\Phi_2 < 0\ (> 0)$, *then flip bifurcation is supercritical (subcritical), the period-2 orbital at* E° *is stable (unstable).*

Proof. Let $\bar{u} = x(n)-1,\ \bar{v} = y(n)-1,\ \bar{w} = z(n)$, system (3.4) moves the ordinary fixed point to the origin to become

$$\begin{pmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{pmatrix} \mapsto \begin{pmatrix} \bar{u}+s_1\Xi_3 \\ \bar{v}+s_0\Xi_4 \\ \bar{w}+s_6\bar{w}(s_7-\bar{w}) \end{pmatrix}, \tag{3.8}$$

here

$$\Xi_3 = -\bar{u}+(s_4-2)\bar{v}+s_2\bar{w}-2\bar{u}\bar{v}+s_2\bar{u}\bar{w}-\bar{v}^2+s_3\bar{w}^2-\bar{u}\bar{v}^2+s_3\bar{u}\bar{w}^2,$$

$$\Xi_4 = \bar{u} + \bar{v} + s_5\bar{w} + 2\bar{u}\bar{v} + s_5\bar{u}\bar{w} + \bar{v}^2 + \bar{u}\bar{v}^2 - s_3\bar{w}^2 - s_3\bar{u}\bar{w}^2.$$

Note

$$\bar{V} == \begin{pmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{pmatrix}, J_0 = \begin{pmatrix} -s_1+1 & s_1(-2+s_4) & s_1s_2 \\ s_0 & s_0+1 & s_0s_5 \\ 0 & 0 & s_6s_7+1 \end{pmatrix},$$

$$\bar{F}(\bar{u},\ \bar{v},\ \bar{w}) = \begin{pmatrix} s_1(-2\bar{u}\bar{v} + s_2\bar{u}\bar{w} - \bar{v}^2 + s_3\bar{w}^2 - \bar{u}\bar{v}^2 + s_3\bar{u}\bar{w}^2) \\ s_0(2\bar{u}\bar{v} + s_5\bar{u}\bar{w} + \bar{v}^2 + \bar{u}\bar{v}^2 - s_3\bar{w}^2 - s_3\bar{u}\bar{w}^2) \\ -s_6\bar{w}^2 \end{pmatrix},$$

that is change to

$$\bar{V} \mapsto J_0\bar{V} + \bar{F}(\bar{u},\ \bar{v},\ \bar{w}) \tag{3.9}$$

Based on the analysis in the previous section, when $s_6 > 0,\ \Phi_0 = 0,\ R_0 \neq 0,\ 2\ (s_1 - s_0 \neq 2,\ 4)$, system (3.9) has a unique eigenvalue at the origin $\mu = -1$. Let its corresponding eigenvector and companion eigenvectors are $q_{04},\ p_{04}$. By calculating $J_0q_{04} = \mu q_{04},\ J_0^T p_{04} = \mu p_{04}$, and $< p_{04},\ q_{04} >= 1$, we get

$$q_{04} = \frac{1}{s_0(s_1-s_0-4)} \begin{pmatrix} s_0+2 \\ -s_0 \\ 0 \end{pmatrix}, \quad p_{04} = \frac{1}{2+s_6s_7} \begin{pmatrix} -s_0(2+s_6s_7) \\ (2-s_1)(2+s_6s_7) \\ s_0(s_1s_5+s_1s_2-2s_5) \end{pmatrix}.$$

According to Reference [24], we decompose $\bar{V} \in \mathbb{R}^3$ into

$$\bar{V} = \varepsilon q_{04} + W.$$

For variable $\varepsilon,\ W$, there is

$$\begin{cases} \varepsilon =< p_{04}, \bar{V} >, \\ W = \bar{V} - < p_{04}, \bar{V} > q_{04}, \end{cases} \tag{3.10}$$

In the coordinate system (3.10), system (3.9) is converted to

$$\begin{cases} \widetilde{\varepsilon} = -\varepsilon + l_0\varepsilon^2 + \varepsilon < p_{04},\ \bar{B}(q_{04},\ W) > + l_1\varepsilon^3 + O(|u|^4), \\ \widetilde{W} = J_0W + \frac{1}{2}l_2\varepsilon^2 + O(|\varepsilon|^3), \end{cases} \tag{3.11}$$

where function $\bar{B},\ \bar{C}$ are the multiple linear functions of function $\bar{F}(\bar{u},\ \bar{v},\ \bar{w})$. We can calculate

$$\bar{B}(q_{04},\ q_{04}) = \frac{2(s_0+4)}{(s_1-s_0-4)^2} \begin{pmatrix} \frac{s_1}{s_0} \\ -1 \\ 0 \end{pmatrix}, \quad \bar{C}(q_{04},\ q_{04},\ q_{04}) = \frac{6(s_0+2)}{s_0(s_1-s_0-4)^3} \begin{pmatrix} -s_1 \\ s_0 \\ 0 \end{pmatrix},$$

thereby getting

$$l_0 = -\frac{2(s_0+4)}{(s_1-s_0-4)^2}, l_1 = \frac{2(s_0+2)}{(s_1-s_0-4)^3},$$

$$l_2 = \frac{2(s_0+4)}{s_0(s_1-s_0-4)^3}\begin{pmatrix} s_1^2 - s_1s_0 - 4s_1 + 2s_0 + 4 \\ s_0^2 + 2s_0 - s_0s_1 \\ 0 \end{pmatrix}.$$

Because $s_0 > 0,\ R_0 = s_1 - s_0 - 2 \neq 2$, so $l_0 \neq 0$. By the central manifold theorem [2], the system exists the central manifold $W_0^c(0,\ 0)$ as shown below

$$W_0^c(0,\ 0) = \{W \in \mathbb{R}^3 | W = \frac{1}{2}w_2\varepsilon^2 + O(|\varepsilon|^3)\},$$

here

$$(J_0 - E)^{-1} = \begin{pmatrix} \frac{1}{s_1(1-s_4)} & \frac{2-s_4}{s_0(1-s_4)} & \frac{-2s_5+s_4s_5-s_2}{s_6s_7(1-s_4)} \\ -\frac{1}{s_1(1-s_4)} & -\frac{1}{s_0(1-s_4)} & \frac{s_2+s_5}{s_6s_7(1-s_4)} \\ 0 & 0 & \frac{1}{s_6s_7} \end{pmatrix},$$

$$\begin{aligned} w_2 &= -(J_0 - E)^{-1}l_3 \\ &= -\frac{2(s_0+4)}{s_0s_1(s_1-s_0-4)^3(1-s_4)}\begin{pmatrix} s_1^2s_4 - s_0s_1s_4 - s_1^2 - 2s_1s_4 + s_0s_1 + 2s_0 + 4 \\ 2(s_1 - s_0 - 2) \\ 0 \end{pmatrix}. \end{aligned}$$

The limit on the central manifold is

$$\widetilde{\varepsilon} = -\varepsilon + \bar{a_0}\varepsilon^2 + \bar{b_0}\varepsilon^3 + O(|\varepsilon|^4),$$

and

$$\bar{a_0} = l_0 = -\frac{2(s_0+4)}{(s_1-s_0-4)^2},$$

$$\begin{aligned} \bar{b_0} &= \frac{1}{6}\langle p_{04},\ \bar{C}(q_{04},\ q_{04},\ q_{04})\rangle - \bar{a_0}^2 - \frac{1}{2}\langle p_{04},\ \bar{B}(q_{04},\ (J_0-E)^{-1}\bar{B}(q_{04},\ q_{04}))\rangle \\ &= \frac{2(s_0+2)}{(s_1-s_0-4)^3} - \frac{4(s_0+4)^2}{(s_1-s_0-4)^4} - \frac{4(s_0+4)}{(s_1-s_0-4)^3} \\ &= -\frac{2}{(s_1-s_0-4)^4}(s_0^2 + s_1s_0 + 6s_0 + 6s_1 + 8), \end{aligned}$$

The critical canonical coefficients are calculated as follows

$$\begin{aligned} \bar{c_0} &= \bar{a_0}^2 + \bar{b_0} \\ &= -\frac{2(s_0+6)}{(s_1-s_0-4)^3}. \end{aligned}$$

These parameters satisfy $s_0 > 0$, $\bar{c_0} \neq 0$, which means that the system satisfies the non-degenerate condition for the occurrence of flip bifurcation, at this time, the bifurcation occurs at the fixed point E°. Note $\Phi_2 = s_1 - s_0 - 4$, if $\Phi_2 < 0$ (> 0), then flip bifurcation is supercritical (subcritical), and the period-2 orbit at E° is stable (unstable).

In summary, we get the theorem (3.3). □

3.3. Neimark-Sacker bifurcation

In this part, we prove that system undergoes Neimark-Sacker bifurcation.

Same as section (3.3), we translate the fixed point E° to the origin, then the system becomes map (3.9). Thus the eigenvalue equation of map (3.9) at the origin is

$$(\lambda - Q_0)(\lambda^2 + R_0\lambda + S_0) = 0,$$

where

$$Q_0 = s_6 s_7 + 1,\ R_0 = s_1 - s_0 - 2,$$
$$S_0 = s_0 - s_1 + s_0 s_1(1 - s_4) + 1,$$
$$\Phi_1 = S_0 - 1 = s_0 - s_1 + s_0 s_1(1 - s_4).$$

The eigenvalues are respectively

$$\mu_{01,\ 02} = -\frac{R_0}{2} \pm \frac{\sqrt{R_0^2 - 4S_0}}{2},\ \mu_{03} = Q_0.$$

Next selecting s_4 as the bifurcation parameter, when $\Phi_1 = 0,\ 0 < s_1 - s_0 < 4$, let's assume that there are polynomials of $s_4 = s_{40},\ s_0,\ s_1$ about $s40$ respectively. Now

$$|\mu_{01,\ 02}| = \sqrt{S_0(s_4)}, \quad \frac{d\sqrt{S_0(s_4)}}{ds_4}\ |_{s_{40}} = -\frac{1}{2}s_0 s_1 \neq 0.$$

Therefore, the system has a cross-sectional condition of Neimark-Sacker bifurcation at the fixed point.

Next, we analyze the satisfaction of its non-degenerate conditions. Considering $s_6 > 0,\ \Phi_1 = 0,\ 0 < s_1 - s_0 < 4$, the eigenvalues of map (3.9) at the origin have a pair of complex eigenvalues on the unit circle $\mu_{01,\ 02} = e^{\pm io}(coso = -\frac{R_0}{2},\ sino = \frac{\sqrt{4-R_0^2}}{2})$, where $o = arccos(-\frac{R_0}{2})$, and $0 < o < \pi$. If $e^{iko} \neq 1(k = 1,\ 2,\ 3,\ 4)$, namely, $R_0 \neq -2,\ 2,\ 1,\ 0$, system satisfies one of the non-degenerate conditions. Also, we analyze the satisfactoriness of another non-degenerate condition.

Let $q_{05},\ p_{05} \in \mathbb{C}^3$ be the complex eigenvector and adjoint eigenvector corresponding to μ_{01}. By calculating $J_0 q_{05} = \mu_{01} q_{04},\ J_0^T p_{05} = \mu_{02} p_{05}$, and $< p_{05},\ q_{05} >= 1$, we obtain

$$q_{05} = r\begin{pmatrix} s_0 + 1 - \mu_{01} \\ -s_0 \\ 0 \end{pmatrix}, \quad p_{05} = \begin{pmatrix} s_0(1 + s_6 s_7 - \mu_{02}) \\ (s_1 - 1 + \mu_{02})(1 + s_6 s_7 - \mu_{02}) \\ s_0(-s_1 s_5 - s_1 s_2 + s_5 - s_5\mu_{02}) \end{pmatrix},$$

where $r = \frac{1}{s_0(1+s_6 s_7-\mu_{01})(s_0-s_1+2-2\mu_{01})}$.

Available from reference [3], system (3.9) can be expressed as

$$\bar{V} \mapsto J_0\bar{V} + \frac{1}{2}\bar{\mathbf{B}}(\bar{V},\ \bar{V}) + \frac{1}{6}\bar{\mathbf{C}}(\bar{V},\ \bar{V},\ \bar{V}), \tag{3.12}$$

where function $\bar{\mathbf{B}},\ \bar{\mathbf{C}}$ are the multiple linear functions of function $\bar{F}(\bar{u},\ \bar{v},\ \bar{w})$. We can calculate

$$\bar{\mathbf{B}}(q_{05},\ q_{05}) = 2s_0 r^2(s_0 + 2 - 2\mu_{01})\begin{pmatrix} s_1 \\ -s_0 \\ 0 \end{pmatrix},$$

$$\bar{\mathbf{B}}(q_{05},\ \bar{q_{05}}) = 2s_0|r|^2(s_0 + 2 - 2coso)\begin{pmatrix} s_1 \\ -s_0 \\ 0 \end{pmatrix},$$

$$\bar{\mathbf{B}}(\bar{q_{05}},\ \bar{q_{05}}) = 2s_0\bar{r}^2(s_0 + 2 - 2\mu_{02})\begin{pmatrix} s_1 \\ -s_0 \\ 0 \end{pmatrix},$$

$$\bar{\mathbf{C}}(q_{05},\ q_{05},\ \bar{q_{05}}) = 2rs_0^2|r|^2(3s_0 + 3 - 2\mu_{01} - \mu_{02})\begin{pmatrix} -s_1 \\ s_0 \\ 0 \end{pmatrix},$$

thus calculated

$$K_{20} = 2s_0^2r^2(s_0 + 2 - 2\mu_{01})(1 + s_6s_7 - \mu_{01})(1 - \mu_{01}),$$

$$K_{11} = 2s_0^2|r|^2(s_0 + 2 - 2coso)(1 + s_6s_7 - \mu_{01})(1 - \mu_{01}),$$

$$K_{02} = 2s_0^2\bar{r}^2(s_0 + 2 - 2\mu_{02})(1 + s_6s_7 - \mu_{01})(1 - \mu_{01}),$$

Because the calculation of K_{21} is complicated, here we only give its expression

$$\begin{aligned} K_{21} = &\langle p_{05},\ \bar{\mathbf{C}}(q_{05},\ q_{05},\ \bar{q_{05}})\rangle + 2\langle p_{05},\ \bar{\mathbf{B}}(q_{05},\ (E - J_0)^{-1}\bar{\mathbf{B}}(q_{05},\ \bar{q_{05}}))\rangle \\ &+ \langle p_{05},\ \bar{\mathbf{B}}(\bar{q_{05}},\ (\mu_{01}^2E - J_0)^{-1}\bar{\mathbf{B}}(q_{05},\ q_{05}))\rangle + \frac{\mu_{02}(1 - 2\mu_{01})}{1 - \mu_{01}}K_{20}K_{11} \\ &- \frac{2}{1 - \mu_{02}}|K_{11}|^2 - \frac{\mu_{01}}{\mu_{01}^3 - 1}|K_{02}|^2. \end{aligned}$$

And $K = Re(\frac{\mu_{02}K_{21}}{2}) - Re(\frac{(1-2\mu_{01})\mu_{02}^2}{2(1-\mu_{02})}K_{20}K_{11}) - \frac{1}{2}|K_{11}|^2 - \frac{1}{4}|K_{02}|^2$. If the mapping (3.9) occurs Neimark-Sacker bifurcation, then it needs to satisfy $K \neq 0$. By literature [7], available

Theorem 3.4. *If parameters satisfy* $NS = \{s_i\ (i = 1, ..., 7)|s_6 > 0,\ \Phi_1 = 0,\ 0 < s_1 - s_0 < 4,\ R_0 \neq 1,\ 0\}$ *and* $K \neq 0$, *system* (3.4) *undergoes Neimark-Sacker bifurcation at the fixed point* E°. *If* $K < 0 (> 0)$, *then an attractive exclusive invariant curve is generated from the fixed point.*

4. Conclusion

Word of mouth and advertising effectiveness are of great significance application value in the commodity economy. This paper studies the consequence of advertising and word-of-mouth effects in consumer behavior model, which is divided into continuous and discrete types for dynamic behavior analysis. According to research/statistics, the continuous model undergoes fold bifurcation, Hopf bifurcation, and degenerate fold-Hopf bifurcation; the discrete model undergoes flip bifurcation and Neimark-Sacker bifurcation. Furthermore, numerical simulations using MATLAB including bifurcation diagrams, time series diagrams and phase diagram.

Bifurcation theories provide a theoretical basis for explaining some economic phenomena and enhancing the effectiveness of merchantsÂ¡Â decision-making. In the consumer market, public recognition of a brand is proportional to the effects of advertising and word-of-mouth. When the influence reaches a certain level, however, the number of consumer groups decelerates.

Based on the theory of Hopf bifurcation, the continuous system undergoes supercritical Hopf bifurcation under certain conditions. This point out that the advertising impacts periodically on consumer behavior, whereas a short-term reduction affect little publics recognition of the brand. At this time, merchants consider the investment tradeoff between advertising and product services, rationally optimize resource allocation from product quality, packaging design, service level, creative advertising, etc., to improve its market share and maximize their profits.

The discrete system undergoes super-critical flip bifurcation and produces a stable period-2 orbit, resulting the analysis of flip bifurcation. This indicates that when advertising effect reaches a certain level, the number of potential consumers and consumers changes periodically over time. In the meanwhile, selling high-end luxury goods, long-term goods and other general needs can allocate resources from product innovation, expanding consumer groups, and so on, so as to maximize benefits.

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References

[1] S. Aslam, E. Jadoon and K. Zaman, *Effect of word of mouth on consumer buying behavior*, Mediterranean Journal of Social Sciences, 2011, 2(3), 497–507.

[2] J. Carr, *Application of center manifold therory*, New York: Springer-Verlag, 1981.

[3] F. Dumortier, R. Roussarie and J. Sotomayor, *Bifurcations of Planar Vector Fields, Nilpotent Singularities and Abelian Integrals, Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 1991.

[4] G. Feichtinger, *Hopf bifurcation in an advertising diffusion model*, Journal of Economic Behavior and Organization, 1992, 17, 401–411.

[5] J. Francisco, *AnÅ¡Âlisis dinÅ¡Âmico del mercado actual y potencial de las organizaciones*, Revista europea de direcciÅ¡Â®n y economÅ¡Âªa de la empresa, 2004, 13(1), 131–140.

[6] P. Giudici and G. Passerone, *Data mining of association structures to model consumer behaviour*, Computational Statistics Data Analysis, 2002, 38, 533–541.

[7] J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical system and bifurcation of vector fields*, New York: Springer-Verlag, 1983.

[8] B.D. Hassard, N.D. Kazarinoff and Y. Wan, *Theory and Application of Hopf Bifurcation*, Cambridge: Cambridge University Press, 1981.

[9] M.W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems and Linear Algebra*, New York, NY: Academic Press, 1974.

[10] H. Jiang and J. Ma, *Effects of internet sales promotion on a differential advertising model*, Discrete Dynamics in Nature and Societ, 2018, 8618146, 11.

[11] Y.A. Kuznetsov, *Elements of applied bifurcation theory*, New York: Springer, 2004, 69–105.

[12] X. Liu and D. Xiao, *Complex dynamics behaviors of a discrete-time predator-prey system*, Chaos Soliton Fract, 2007, 32, 80–94.

[13] J.L. Ren and X.P. Li, *Bifurcations in a seasonally forced predator-prey model with generalized Holling type IV functional response*, International Journal of Bifurcation and Chaos, 2016, 26(12), 1650203.

[14] J.L. Ren and L.P. Yu, *Codimension-two bifurcation, chaos and control in a discrete-time information diffusion model*, Journal of Nonlinear Science 2016, 26, 1895–1931.

[15] J.L. Ren and Q.G. Yuan, *Bifurcations of a periodically forced microbial continuous culture model with restrained growth rate*, Chaos, 2017, 27, 083124.

[16] J.L. Ren, D.D. Zhu, and H.Y. Wang, *Spreading-vanishing dichotomy in information diffusion in online social networks with intervention*, Discrete and Continuous Dynamical Systems Series B, 2019, 24(4), 1843–1865.

[17] N. Sirghi and M. Neamtu, *Deterministic and stochastic advertising diffusion model with delay*, Wseas Transactions on Systems and Control, 2013, 8(4), 141–150.

[18] Y.T. Sun, X.X. Guo and J.L. Ren, *What are the factors that determine the number of people coming to China?*, Journal of Nonlinear Modeling and Analysis, 2019, 1(3), 1–12.

[19] D.D. Tang, S.R. Zhang and J.L. Ren, *Dynamics of a general jerky equation*, Journal of Vibration and Control, 2019, 25(4), 922–932.

[20] Y.W. Tao, X.P. Li and J.L. Ren, *A repeated yielding model under periodic perturbation*, Nonlinear Dynamics, 2018, 94, 2511–2525.

[21] G. Tigan, J. Llibre and L. Ciurdariu, *Degenerate fold–Hopf bifurcations in a Rossler–type system*, International Journal of Bifurcation and Chaos, 2017, 27(5), 1750068.

[22] H.Y. Wang, F. Wang, and K. Xu, *Diffusive logistic model towards predicting information diffusion in online social networks*, Proceeding of international conference on distributed computing systems workshops, 2012, 32, 133–139.

[23] C. Yang and C. Zhang, *Modeling correlated effect and random effect of advertising and Word-of-Mouth on product diffusion*, Institute of Electrical and Electronics Engineers, 2015, 15, 1530–1605.

[24] K. Yuri, *Elements of applied bifurcation theory*, New York: Springer, 2004, 69–105.