Sufficient Conditions of Blow-up and Bound Estimations of Blow-up Time for a Parabolic Equation in Multi-dimensional Space*

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Abstract In this paper, we establish some sufficient conditions on the heat source function and the heat conduction function of the parabolic equation to guarantee that \( u(x, t) \) blows up at finite time, and give upper and lower bounds of the blow-up time in multi-dimensional space.

Keywords Sufficient conditions, blow-up, upper and lower bounds, multi-dimensional space.


1. Introduction

In this paper, we deal with the initial-boundary value problem

\[
\begin{align*}
  u_t - \Delta u &= f(u), & (x, t) &\in \Omega \times (0, t^*), \\
  u &= 0, & (x, t) &\in \Gamma_0 \times (0, t^*), \\
  \frac{\partial u}{\partial n} &= g(u), & (x, t) &\in \Gamma_1 \times (0, t^*), \\
  u(x, 0) &= u_0(x) \geq 0, & x &\in \Omega,
\end{align*}
\]

where \( \Omega \) is a bounded domain of \( \mathbb{R}^N (N \geq 2) \) with smooth boundary \( \Gamma := \partial \Omega, \Gamma = \Gamma_0 \cup \Gamma_1, \text{mes} (\Gamma_0 \cap \Gamma_1) = 0, \text{mes} (\Gamma_0) \geq 0, \text{mes} (\Gamma_1) > 0 \) and \( n = (n_1, n_2, \ldots, n_N) \) is the unit outward normal vector on \( \Gamma_1, u_0 \in C^1(\overline{\Omega}), u_0(x) \geq 0, u_0 \not\equiv 0, \) and \( t^* \) is the blow-up time if blow-up occurs. From the physical standpoint, \( f \) is the heat source function and \( g \) is the heat conduction function transmitting into interior of \( \Omega \) from the boundary \( \Gamma_1 \).

The blow-up phenomena of solutions to evolution partial differential equations has received considerable attentions in recent years. For the work in this area, the reader can refer to the book Quittner [9] and papers [1, 4]. Many methods have been used to determine the blow-up of solutions and to indicate an upper bound of the blow-up time. To our knowledge, the first work on lower bound of \( t^* \) was given by Weissler [10, 11]. Recently, a number of papers deriving lower bound of \( t^* \) in various problems have appeared (see [2, 3, 6–8, 12, 13] and the references therein).

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The blow-up for nonlinear equations with Neumann boundary conditions has received considerable attentions. Payne and Schaefer [6] considered homogeneous equation without heat source term
\[ u_t = \Delta u, \quad (x, t) \in \Omega \times (0, t^*). \]
Under suitable nonlinear conditions, they deduced a lower bound of the blow-up time when blow-up occurs only in three-dimensional space. Mizoguchi [5] studied the semilinear heat equation with a power function heat source term
\[ u_t = \Delta u + u^p, \quad (x, t) \in \Omega \times (0, T), \]
and showed that if \( u \) blows up at \( t = T \), then \( |u(t)|_\infty \leq C(T - t)^{-\frac{1}{p-1}} \) for some \( C > 0 \). Payne etc [8] considered heat equation with general heat source term
\[ u_t = \Delta u - f(u) \quad x \in \Omega, t \in (0, t^*), \]
and established conditions on nonlinearities to guarantee that \( u(x, t) \) blows up at some finite time \( t^* \). Moreover, an upper bound for \( t^* \) was derived. Under some more restrictive conditions, a lower bound for \( t^* \) was derived only in three-dimensional space. Li and Li [2] investigated nonhomogeneous divergence form parabolic equation
\[ u_t = \sum_{i=1}^{N} (a_{ij}(x)u_{x_i})_{x_j} - f(u), \quad t \in (0, t^*), \quad x = (x_1, x_2, \cdots, x_N) \in \Omega, \]
and gave the conditions on nonlinearities to guarantee that \( u(x, t) \) exists globally or blows up at some finite time respectively. If blow-up occurs, they obtained upper and lower bounds of the blow-up time, but the lower bound of \( t^* \) was valid only in three-dimensional space.

Motivated by the above work, we intend to study the blow-up phenomena for problem (1.1). It is well known that the data \( f \) and \( g \) may greatly affect the behavior of \( u(x, t) \) with the development of time. The larger the heat source function \( f \) and conduction function \( g \) are, the greater possibility the blow-up will occur, and the earlier blow-up time will be. The main contributions of this paper are: (i) the conditions of blow-up are derived naturally by means of calculation process, and some examples satisfying the conditions are given; (ii) the lower bound of blow-up time is given under the conditions that ensure occurrence of blow-up phenomena; (iii) the lower bound of blow-up time is obtained in multi-dimensional space which improves the situation discussed in three-dimensional space.

The present work is organized as follows. In Section 2, we derive the conditions on \( f, g \) to ensure that the solutions blow up at finite time and obtain an upper bound of the blow-up time. In Section 3, under the conditions on \( f \) and \( g \) that guarantee the occurrence of blow-up, we get a lower bound of blow-up time \( t^* \) in multi-dimensional space.

2. Blow-up and upper bound estimation of \( t^* \)

In order to derive the sufficient conditions for blow-up phenomena and the upper bound of blow-up time, we first give the following calculation. From the physical
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background and characteristics of the equation, we know that if the functions $f, g$ are nonnegative, then the solution to (1.1) is nonnegative and smooth.

Multiplying the equation of (1.1) by $u$ and integrating on $\Omega$, we obtain

$$ \int_{\Omega} u u_t \, dx = \int_{\Omega} u \Delta u \, dx + \int_{\Omega} u f(u) \, dx, $$

that is

$$ \frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \, dx = \int_{\Gamma_1} u \frac{\partial u}{\partial n} \, dS - \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} u f(u) \, dx. $$

(2.1)

These calculations inspire us to define

$$ \Phi(t) := \frac{1}{2} \int_{\Omega} u^2 \, dx. $$

If

$$ u g(u) \geq \gamma G(u), \quad u f(u) \geq \gamma F(u), $$

(2.2)

where

$$ G(\xi) = \int_{0}^{\xi} g(s) \, ds, \quad F(\xi) = \int_{0}^{\xi} f(s) \, ds, \quad \gamma \geq 2, $$

(2.3)

then, from (2.1), we have

$$ \Phi'(t) = \int_{\Gamma_1} u g(u) \, dS - \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} u f(u) \, dx $$

$$ \geq \gamma \left[ \int_{\Gamma_1} G(u) \, dS - \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} F(u) \, dx \right]. $$

(2.4)

Denote

$$ \Theta(t) := \int_{\Gamma_1} G(u) \, dS - \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} F(u) \, dx, $$

then with (2.4) we see

$$ \Phi'(t) \geq \gamma \Theta(t). $$

(2.5)

Since

$$ \Theta'(t) = \int_{\Gamma_1} g(u) u_t \, dS - \frac{1}{2} \int_{\Omega} (|\nabla u|^2)_t \, dx + \int_{\Omega} f(u) u_t \, dx, $$

(2.6)

and noting

$$ \frac{1}{2} (|\nabla u|^2)_t = \text{div}(u_t \nabla u) - u_t \Delta u, $$

by using divergence theorem, we get that

$$ \frac{1}{2} \int_{\Omega} ((|\nabla u|^2)_t) \, dx = \int_{\Omega} \text{div}(u_t \nabla u) - u_t \Delta u \, dx. $$
\[
\begin{align*}
\int_{\Gamma_1} u_t \frac{\partial u}{\partial n} dS - \int_{\Omega} u_t \Delta u dx &= \int_{\Gamma_1} u_t g(u) dS - \int_{\Omega} u_t \Delta u dx. \\
\end{align*}
\] (2.7)

Substituting (2.7) into (2.6), we have
\[
\Theta'(t) = \int_{\Omega} u_t [\Delta u + f(u)] dx = \int_{\Omega} u_t^2 dx \geq 0. \\
\] (2.8)

Assuming
\[
\Theta(0) = \int_{\Gamma_1} G(u_0) dS - \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 dx + \int_{\Omega} F(u_0) > 0, \\
\] (2.9)

we have \(\Theta(t) > 0\) for any \(t \in (0, t^*)\). Using the definition of \(\Phi(t)\), (2.8) and Hölder's inequality, we have
\[
(\Phi'(t))^2 = \left( \int_{\Omega} u u_t dx \right)^2 \leq \left( \int_{\Omega} u^2 dx \right) \left( \int_{\Omega} u_t^2 dx \right) = 2 \Phi(t) \Theta'(t). \\
\] (2.10)

Combining (2.10) and (2.5), we obtain
\[
\Phi(t) \Theta'(t) \geq \frac{1}{2} (\Phi'(t))^2 \geq \frac{\gamma}{2} \Phi'(t) \Theta(t),
\]
that is
\[
\Phi(t) \Theta'(t) \geq \frac{\gamma}{2} \Phi'(t) \Theta(t). \\
\] (2.11)

Multiplying the both sides of (2.11) by \(\Phi^{-1/\gamma + 1}\), we deduce
\[
\left( \Theta(t) \Phi^{-1/\gamma + 1} (t) \right)' = \Phi^{-1/\gamma + 1} (t) \left( \Phi(t) \Theta'(t) - \frac{\gamma}{2} \Phi'(t) \Theta(t) \right) \geq 0. \\
\] (2.12)

Integrating (2.12) over \([0, t]\) yields
\[
\Theta(t) \Phi^{-1/\gamma + 1} (t) = \Theta(0) \Phi^{-1/\gamma + 1} (0) =: M,
\]
that is
\[
\Theta(t) \geq M \Phi^{-1/\gamma + 1} (t). \\
\] (2.13)

By (2.5) and (2.13), we get
\[
\Phi'(t) \geq \gamma \Theta(t) \geq \gamma M \Phi^{-1/\gamma + 1} (t),
\]
which implies (if \(\gamma > 2\)) that
\[
\Phi^{-1/\gamma + 1} (t) \geq \gamma M \Rightarrow \frac{2}{2 - \gamma} \left( \Phi^{-1/\gamma + 1} (t) \right)' \geq \gamma M \\
\Rightarrow \left( \Phi^{-1/\gamma + 1} (t) \right)' \leq \frac{2 - \gamma}{2} \gamma M
\]
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\[ \Rightarrow \Phi^{-\frac{2}{\gamma}+1}(t) \leq \Phi^{-\frac{2}{\gamma}+1}(0) + \frac{2-\gamma}{2} \gamma Mt \]

\[ \Rightarrow \Phi^{\frac{2}{\gamma}-1}(t) \geq \frac{1}{\Phi^{-\frac{2}{\gamma}+1}(0) - \frac{2-\gamma}{2} \gamma Mt} \]

\[ \Rightarrow \Phi(t) \geq \frac{1}{[\Phi^{-\frac{2}{\gamma}+1}(0) - \frac{2-\gamma}{2} \gamma Mt]^{\frac{2}{\gamma-2}}} \quad (2.15) \]

Therefore,

\[ \lim_{t \to t^*} \Phi(t) = +\infty, \quad (2.16) \]

where \( t^* \leq T = \frac{2\Phi(0)}{\gamma(\gamma-2)\Theta(0)} \) (by the definition of \( M \)).

Summarizing the above conditions and processes, we can get the following theorem.

**Theorem 2.1.** Supposed that \( \Omega \subset \mathbb{R}^N \quad (N \geq 2) \) is a bounded domain, and the nonnegative integrable functions \( f \) and \( g \) satisfy the conditions for some constant \( \gamma > 2 \):

\[ \xi f(\xi) \geq \gamma F(\xi), \quad \xi g(\xi) \geq \gamma G(\xi), \quad \forall \xi \geq 0, \]

with

\[ F(\xi) := \int_0^\xi f(s)ds, \quad G(\xi) := \int_0^\xi g(s)ds. \]

Moreover, we assume \( \Theta(0) > 0 \) with

\[ \Theta(t) = \int_{\Gamma_1} G(u)dS - \frac{1}{2} \int_{\Omega} |\nabla u|^2d\mathbf{x} + \int_{\Omega} F(u)d\mathbf{x}. \]

Then the nonnegative classical solution \( u(x,t) \) of problem (1.1) blows up at time \( t^* \leq T \) with

\[ T = \frac{2\Phi(0)}{\gamma(\gamma-2)\Theta(0)}, \]

where \( \Phi(t) = \frac{1}{2} \int_{\Omega} u^2d\mathbf{x} \).

**Remark 2.1.** (1) If we choose \( f(u) = u^\alpha, \quad g(u) = u^\beta, \quad (\alpha, \beta > 1), \quad u_0(x) = \text{constant} > 0 \), then all the conditions in the theorem are satisfied.

(2) When the equation has no heat source, that is \( f \equiv 0 \), we can choose \( g(u) = u^\beta, \quad (\beta > 1), \quad u_0(x) = \text{constant} > 0 \), then all the conditions in the theorem are also satisfied. This situation shows that blow-up phenomena only depending on boundary heat conduction may also occur, but the blow-up time will be delayed.

(3) When the boundary is adiabatic, that is \( g \equiv 0 \), we can choose \( f(u) = u^\alpha, \quad (\alpha > 1), \quad u_0(x) = \text{constant} > 0 \), then all the conditions in the theorem are also satisfied. This situation shows that blow-up phenomena only depending on heat source may also occur, but the blow-up time will be delayed.

(4) When the equation has no heat source and the boundary is adiabatic, that is \( f \equiv 0 \) and \( g \equiv 0 \), from (2.1), we know that the energy functional \( \Phi(t) \) is decreasing, so blow-up phenomena will not occur.
Remark 2.2. Our theorem can be used to explain the results for some problems in literatures. For example, in [12,13], the authors dealt with a heat equation with a local nonlinear Neumann boundary conditions:

\[
\begin{aligned}
\begin{cases}
u_t = \Delta u, & \text{in } \Omega \times (0,t^*), \\
\frac{\partial u}{\partial n} = u^q, & \text{on } \Gamma_1 \times (0,t^*), \\
\frac{\partial u}{\partial n} = 0, & \text{on } \Gamma_2 \times (0,t^*), \\
u(x,0) = u_0(x), & \text{in } \Omega,
\end{cases}
\end{aligned}
\]

where \( q > 1, \ u_0 \in C^1(\overline{\Omega}), \ u_0(x) \geq 0, \) and \( u_0(x) \neq 0. \)

In fact, for \( q > 1, \) there exists a constant \( \gamma > 1 \) such that \( \frac{\gamma}{1 + q} \leq 1, \) that is

\[
u g(u) = u^{q+1} \geq \gamma \int_0^u s^q ds = \frac{\gamma}{1 + q} u^{q+1}.
\]

One can choose a suitable \( u_0(x) \) such that all the conditions in the Theorem 2.1 are satisfied. Consequently, the blow-up phenomena occurs.

3. Lower bound estimation of \( t^* \)

Lemma 3.1. Let \( \Omega \subset \mathbb{R}^N \) \( (N \geq 2) \) be a bounded star-shaped domain assumed to be convex in \( N - 1 \) orthogonal directions. Then for any nonnegative increasing \( C^1 \) function \( P(w), \) we have

\[
\int_{\partial \Omega} P(w) dS \leq \frac{N}{\rho_0} \int_{\Omega} P(w) dx + \frac{d}{\rho_0} \int_{\Omega} P'(w)|\nabla w| dx,
\]

where

\[
\rho_0 := \min_{x \in \partial \Omega} (x \cdot n), \quad d := \max_{x \in \partial \Omega} |x|.
\]

Proof. Since \( \Omega \) is a bounded star-shaped domain, we have \( \rho_0 > 0. \) Integrating the identity

\[
\text{div}(P(w)x) = NP(w) + P'(w)(x \cdot \nabla w)
\]

over \( \Omega, \) and using the divergence theorem, we get

\[
\int_{\partial \Omega} P(w)(x \cdot n) dS = N \int_{\Omega} P(w) dx + \int_{\Omega} P'(w)(x \cdot \nabla w) dx.
\]

By the definitions of \( \rho_0 \) and \( d, \) it follows that

\[
\rho_0 \int_{\partial \Omega} P(w) dS \leq \int_{\partial \Omega} P(w)(x \cdot n) dS
\]

\[
\leq N \int_{\Omega} P(w) dx + \int_{\Omega} P'(w)|x|\nabla w| dx
\]

\[
\leq N \int_{\Omega} P(w) dx + d \int_{\Omega} P'(w)|\nabla w| dx,
\]

which implies the desire conclusion.
Lemma 3.2. Assume that \( \Omega \subset \mathbb{R}^N (N \geq 3) \) is a bounded star-shaped domain assumed to be convex in \( N-1 \) orthogonal directions. Let \( w(x) \) be a nonnegative \( C^1 \) function defined in \( \Omega \). Then for any constant \( \sigma \geq 1 \), the following inequality holds

\[
\int_{\Omega} w^{(1+\frac{2}{N-2})\sigma} \, dx \\
\leq (1+2d)^{N-3} \left[ \frac{N}{2\rho_0} \int_{\Omega} w^{\sigma} \, dx + \frac{\sigma}{2} \left( 1 + \frac{d}{\rho_0} \right) \int_{\Omega} w^{\sigma-1} |\nabla w| \, dx \right]^{1+\frac{1}{2}}.
\]

where

\[
\rho_0 := \min_{x \in \partial \Omega} (x \cdot n) > 0, \quad d := \max_{x \in \partial \Omega} |x|.
\]

Proof. Using mathematical induction method and iterated integral formula, we can finish the proof.

In this section, in the multi-dimensional space, we give the lower bound of blow-up time under the conditions that guarantee the occurrence of blow-up phenomena. We assume that functions \( f \) and \( g \) satisfying

\[
f(\xi) \equiv 0, \quad \gamma G(\xi) \leq \xi g(\xi) \leq \tau \xi^{2+\frac{2}{N-2}}, \quad \xi \geq 0, \quad \gamma > 2 \quad (3.1)
\]

for \( \tau > 0 \). For

\[
\varphi(t) := \frac{1}{2} \int_{\Omega} u^2 \, dx, \quad (3.2)
\]

we can show that \( \varphi(t) \) satisfies

\[
\varphi'(t) \leq \Psi(\varphi) \quad (3.3)
\]

for some computable function \( \Psi \). Then it follows that \( t^* \) is bounded by

\[
t^* \geq \int_{\varphi(0)}^{\infty} \frac{d\eta}{\Psi(\eta)}. \quad (3.4)
\]

Indeed, differentiating (3.2), we have (noting \( f \equiv 0 \))

\[
\varphi'(t) = \int_{\Omega} u u_t \, dx = \int_{\Omega} u [\Delta u + f(u)] \, dx = \int_{\Gamma_1} u g(u) \, dS - \int_{\Omega} |\nabla u|^2 \, dx \\
\leq \tau \int_{\Gamma_1} u^{2+\frac{2}{N-2}} \, dS - \int_{\Omega} |\nabla u|^2 \, dx. \quad (3.5)
\]

Using Lemma 3.1, we have

\[
\int_{\Gamma_1} u^{2+\frac{2}{N-2}} \, dS \leq \int_{\partial \Omega} u^{2+\frac{2}{N-2}} \, dS \\
\leq \frac{N}{\rho_0} \int_{\Omega} u^{2+\frac{2}{N-2}} \, dx + \frac{(2^{N-1}+1)d}{2^{N-2}\rho_0} \int_{\Omega} u^{1+\frac{2}{N-2}} |\nabla u| \, dx. \quad (3.6)
\]

Using H"older’s and Young’s inequalities, we get

\[
\int_{\Omega} u^{2+\frac{2}{N-2}} \, dx \leq \frac{1}{2} \int_{\Omega} u^{2+\frac{2}{N-2}} \, dx + \frac{1}{2} \int_{\Omega} u^2 \, dx.
\]
Combining (3.9)-(3.11), we obtain
\begin{equation}
\varphi(t) \leq \frac{N\tau}{\rho_0} \varphi(t) + \frac{\tau}{2\rho_0} \left( N + \frac{(2N-1)d}{2N-2} \right) \int_{\Omega} u^{2+\frac{1}{\nu-\tau}} \|\nabla u\|^{1+\frac{1}{\nu-\tau}} \\
+ \left( \frac{(2N-1)\mu \tau d}{2N-1 \rho_0} - 1 \right) \int_{\Omega} |\nabla u|^2 dx. \tag{3.9}
\end{equation}

Applying Lemma 3.2 and Hölder’s inequality, we have
\begin{equation}
\int_{\Omega} u^{2+\frac{1}{\nu-\tau}} dx = \int_{\Omega} u^{(1+\frac{1}{\nu-\tau})^2} dx \\
\leq (1 + 2d)^{N-3} \left( \frac{N}{2\rho_0} \int_{\Omega} u^2 dx + \left( 1 + \frac{d}{\rho_0} \right) \int_{\Omega} u |\nabla u| dx \right)^{1+\frac{1}{\nu-\tau}} \\
\leq (1 + 2d)^{N-3} \left( \frac{N}{2\rho_0} \int_{\Omega} u^2 dx + \left( 1 + \frac{d}{\rho_0} \right) \left( \int_{\Omega} u^2 dx \int_{\Omega} |\nabla u|^2 dx \right)^\frac{1}{2} \right)^{1+\frac{1}{\nu-\tau}} \\
\leq C(1 + 2d)^{N-3} \left[ \left( \frac{N}{\rho_0} \right)^{1+\frac{1}{\nu-\tau}} \varphi^{1+\frac{1}{\nu-\tau}}(t) \\
+ \left( 1 + \frac{d}{\rho_0} \right)^{1+\frac{1}{\nu-\tau}} 2^{\frac{1}{2+\frac{1}{\nu-\tau}}} \varphi^{\frac{1}{2+\frac{1}{\nu-\tau}}}(t) \left( \int_{\Omega} |\nabla u|^2 dx \right)^\frac{1}{2+\frac{1}{\nu-\tau}} \right] \tag{3.10}
\end{equation}
for some constant $C$. Using Young’s inequality with $\varepsilon$, we have
\begin{equation}
\varphi^{\frac{1}{2+\frac{1}{\nu-\tau}}}(t) \left( \int_{\Omega} |\nabla u|^2 dx \right)^\frac{1}{2+\frac{1}{\nu-\tau}} \\
= \left( \varepsilon^{2N-2+1} \left( \int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2+\frac{1}{\nu-\tau}}} \right) \varepsilon^{-\frac{2N-2+1}{2N-1}} \varphi^{\frac{1}{2+\frac{1}{\nu-\tau}}}(t) \\
\leq \frac{2N-2 + 1}{2N-1} \varepsilon \int_{\Omega} |\nabla u|^2 dx + \frac{2N-2 - 1}{2N-1} \varepsilon^{-\frac{2N-2+1}{2N-2-1}} \varphi^{\frac{2N-2+1}{2N-2-1}}(t). \tag{3.11}
\end{equation}
Combining (3.9)-(3.11), we obtain
\begin{equation}
\varphi'(t) \leq c_1 \varphi + c_2 \varphi^{\frac{2N-2+1}{2N-2-1}} + c_3 \varphi^{\frac{2N-2+1}{2N-2-1}} + c_4 \int_{\Omega} |\nabla u|^2 dx, \tag{3.12}
\end{equation}
where
\begin{equation}
c_1 = \frac{N\tau}{\rho_0},
\end{equation}
and
\begin{equation}
\int_{\Omega} u^{1+\frac{1}{\nu-\tau}} |\nabla u| dx \leq \frac{1}{2\mu} \int_{\Omega} u^{2+\frac{1}{\nu-\tau}} dx + \frac{\mu}{2} \int_{\Omega} |\nabla u|^2 dx, \tag{3.8}
\end{equation}
for all $\mu > 0$.

Inserting (3.6)-(3.8) into (3.5), we obtain
Choosing $\varepsilon$ small enough, one can get a positive $\mu$ such that $c_4 = 0$. Therefore,
\[ \varphi'(t) \leq c_1 \varphi + c_2 \varphi^{\frac{2N-2+1}{2N-2}} + c_3 \varphi^{\frac{2N-2+1}{2N-2}} := \Psi(\varphi). \] (3.13)

From (3.13), we get
\[ \left( \int_{\varphi(0)}^{\varphi(t)} \frac{d\eta}{\Psi(\eta)} \right)' = \frac{\varphi'(t)}{\Psi(\varphi)} \leq 1. \] (3.14)

Integrating (3.14) over $[0, t]$, we obtain
\[ t \geq \int_{\varphi(0)}^{\varphi(t)} \frac{d\eta}{\Psi(\eta)}, \] (3.15)

which implies
\[ t^* \geq \int_{\varphi(0)}^{\infty} \frac{d\eta}{\Psi(\eta)} = \int_{\varphi(0)}^{\infty} \frac{d\eta}{c_1 \eta + c_2 \eta^{\frac{2N-2+1}{2N-2}} + c_3 \eta^{\frac{2N-2+1}{2N-2}}}, \]

with $\lim_{t \to t^*} \varphi(t) = \infty$ (by Theorem 2.1).

From the above analysis and Theorem 2.1, we can summarize the following theorem on lower bound estimation of blow-up time $t^*$:

**Theorem 3.1.** Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be a bounded star-shaped domain assumed to be convex in $N - 1$ orthogonal directions, and the nonnegative $f$ and $g$ satisfy the conditions
\[ f(\xi) \equiv 0, \quad \gamma G(\xi) \leq \xi g(\xi) \leq \tau \xi^{2+\frac{1}{N-1}}, \quad \xi \geq 0, \quad \gamma > 2. \]

Then the nonnegative solution $u(x, t)$ of problem (1.1) blows up at finite time, and the blow-up time $t^*$ is bounded from below by
\[ t^* \geq \int_{\varphi(0)}^{\infty} \frac{d\eta}{\Psi(\eta)} = \int_{\varphi(0)}^{\infty} \frac{d\eta}{c_1 \eta + c_2 \eta^{\frac{2N-2+1}{2N-2}} + c_3 \eta^{\frac{2N-2+1}{2N-2}}}. \]

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