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Oscillation Results for BVPs of Even Order Nonlinear Neutral Partial Differential Equations*

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Abstract A class of boundary value problems (BVPs) of even order neutral partial functional differential equations with continuous distribution delay and nonlinear diffusion term are studied. By applying the integral average and Riccati's method, the high-dimensional oscillatory problems are changed into the one-dimensional ones, and some new sufficient conditions are obtained for oscillation of all solutions of such boundary value problems under first boundary condition. The results generalize and improve some results of the latest literature.

Keywords Even order partial functional differential equation, boundary value problem, oscillation criteria, continuous distribution delay, nonlinear diffusion term.

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1. Introduction

The oscillation study of partial functional differential equations (PFDE) are of both theoretical and practical interest. Some applicable examples in such fields as population kinetics, chemistry reactors and control system can be found in the monograph of Wu [9]. There have been some results on the oscillations of solutions of various types of PFDE. Here, we mention the literatures of Kiguradze, Kusano and Yoshida [2], Thandapani and Savithri [8], Saker [5], Li and Debnath [3], Wang and Wu [10], Yang [12], Wang, Wu and Caccetta [11], ShouKaKu [6], ShouKaKu, Stavroulakis and Yoshida [7] and the references cited therein. To the best of our knowledge, there are fewer to investigate the oscillation of solutions of PFDE with continuous distribution delay. However, we note that in many areas of their actual

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application, models describing these problems are often effected by such factors as seasonal changes. Therefore it is necessary, either theoretically or practically, to study a type of PFDE in a more general sense–PFDE with continuous distribution delay. In this paper, we will discuss the oscillation of solutions of the high-order neutral partial functional differential equations with continuous distribution delay and nonlinear diffusion term

$$\frac{\partial^n}{\partial t^n}[u+\int_c^d p(t,\eta)u[x,r(t,\eta)]d\tau(\eta)]+\int_a^b f(x,t,\xi,u[x,g(t,\xi)])d\mu(\xi)$$
$$=a_0(t)h_0(u)\Delta u+a_1(t)h_1(u(x,\sigma(t)))\Delta u(x,\sigma(t)),\quad (t,x)\in\Omega\times R_+\equiv G,\quad (1.1)$$

where $n\geq 2$ is even, Ω is a bounded domain in R^m with a piecewise smooth boundary $\partial\Omega$, Δ is the Laplacian in R^m, $R_+=[0,\infty)$, the integral of Eq.(1.1) are Stieltjes ones.

Consider first boundary condition:

$$u(x,t)=0,\quad (x,t)\in\partial\Omega\times R_+.\quad (1.2)$$

Throughout this paper, assume that the following conditions hold:

(H_1) $p(t,\eta)\in C(I\times[c,d],R)$, $I=[t_0,\infty)$, $t_0\in R$, $p(t,\eta)\geq 0$, $P(t)=\int_c^d p(t,\eta)d\tau(\eta)\leq P<1$, P is a constant;

(H_2) $r(t,\eta)\in C(I\times[c,d],R)$, $r(t,\eta)\leq t$, $\lim_{t\to\infty}\min_{\eta\in[c,d]}r(t,\eta)=\infty$;

(H_3) $g(t,\xi)\in C(I\times[a,b],R)$ is nondecreasing with respect to t and ξ, respectively, $\frac{d}{dt}g(t,a)$ exists, $g(t,\xi)\leq t$ for $\xi\in[a,b]$, $\lim_{t\to\infty}\min_{\xi\in[a,b]}g(t,\xi)=\infty$;

(H_4) $a_0(t)$, $a_1(t)\in C(I,R_+)$, $\sigma(t)\in C(I,R)$, $\lim_{t\to\infty}\sigma(t)=\infty$;

(H_5) $h_0(u)$, $h_1(u)\in C^1(R,R)$, $uh_0'(u)\geq 0$, $uh_1'(u)\geq 0$, $h_0(0)=0$, $h_1(0)=0$;

(H_6) $f(x,t,\xi,u)\in C(\Omega\times R_+\times[a,b]\times R_+,R)$;

(H_7) $\tau(\eta)$, $\mu(\xi)$ is nondecreasing on $[c,d]$ and $[a,b]$, respectively.

Definition 1.1. A function $u(x,t)\in C^n(G)\cap C^1(\overline{G})$ is said to be a solution of the boundary value problems (1.1), (1.2) if it satisfies (1.1) in G and boundary condition (1.2) in $\partial\Omega\times R_+$.

Definition 1.2. A solution $u(x,t)$ of the boundary value problems (1.1), (1.2) is said to be oscillatory in G if it has arbitrarily large zeros, namely, for any $T>0$, there exists a point $(x_1,t_1)\in\Omega\times[T,\infty)$ such that the equality $u(x_1,t_1)=0$ holds. Otherwise, the solution $u(x,t)$ is called nonoscillatory in G.

The objective of this paper is to derive some new oscillatory criteria of solutions of the boundary value problems (1.1), (1.2). It should be noted that in the proof we do not use the results of Dirichlet's eigenvalue problem.

To prove the main results of this paper, we need the following lemmas.

Lemma 1.1 (Kiguradze [1])**.** *Let $y(t)\in C^n(I,R)$ be of constant sign, $y^{(n)}(t)\neq 0$ and $y^{(n)}(t)y(t)\leq 0$ on I, then*

(i) there exists a $t_1\geq t_0$, such that $y^{(i)}(t)(i=1,2,\cdots,n-1)$ is of constant sign on $[t_1,\infty)$;

(ii) there exists an integer $l\in\{0,1,2,\cdots,n-1\}$, with $n+l$ odd, such that

$$y^{(i)}(t)>0,\ t\geq t_1,\ i=0,1,2,\cdots,l;$$
$$(-1)^{i+l}y^{(i)}(t)>0,\ t\geq t_1,\ i=l+1,\cdots,n.$$

.

Lemma 1.2 (Philos [4]). *Suppose that $y(t)$ satisfies the conditions of Lemma 1 and $y^{(n-1)}(t)y^{(n)}(t) \leq 0,\ t \geq t_1$, then for every $\theta \in (0,1)$, there exists a constant $N > 0$ satisfying*

$$|y'(\theta t)| \geq Nt^{n-2}|y^{(n-1)}(t)|,\ t \geq t_1.$$

2. Main results

Let $u(x,t)$ be a solution of the boundary value problems (1.1), (1.2), we definite

$$V(t) = (\int_\Omega dx)^{-1} \int_\Omega u(x,t)dx. \tag{2.1}$$

Theorem 2.1. *Suppose that there exist $q(t,\xi) \in C([t_0,\infty) \times [a,b], R_+)$ and $F(u) \in C(R,R)$, $F(u)$ is a lower convex function on $(0,\infty)$, such that*

$$f(x,t,\xi,u)sgnu \geq q(t,\xi)F(u)sgnu, \tag{2.2}$$

$$-F(-u) \geq F(u) \geq Mu > 0\ (u > 0,\ \textit{and } M \textit{ is a positive constant}). \tag{2.3}$$

If there exists a function $\rho(t) \in C^1(I, R_+)$, such that

$$\int_0^\infty [\lambda M\rho(t)Q(t) - \frac{(\rho'(t))^2}{4\lambda N g^{n-2}(t,a)g'(t,a)\rho(t)}]dt = \infty, \tag{2.4}$$

where $Q(t) = \int_a^b q(t,\xi)d\mu(\xi)$, $\lambda = 1 - P$, P is defined by (H_1), then all solutions of the boundary value problems (1.1), (1.2) *are oscillatory in G.*

Proof. Suppose that there exists a nonoscillatory solution $u(x,t)$ of the boundary value problems (1.1), (1.2). Without loss of generality, we may assume that $u(x,t) > 0$ in $\Omega \times I$, then form (H_2), (H_3), (H_4), there exists a $t_1 \geq t_0$, such that $u[x,r(t,\eta)] > 0,\ u[x,g(t,\xi)] > 0,\ u(x,\sigma(t)) > 0,\ (x,t) \in \Omega \times [t_1,\infty),\ \eta \in [c,d],\ \xi \in [a,b]$.

Integrating both sides of (1.1) with respect to x over the domain G, we have

$$\begin{aligned}
&\frac{d^n}{dt^n}[\int_\Omega udx + \int_\Omega \int_c^d p(t,\eta)u[x,r(t,\eta)]d\tau(\eta)dx] \\
=&a_0(t)\int_\Omega h_0(u)\Delta udx + a_1(t)\int_\Omega h_1(u(x,\sigma(t)))\Delta u(x,\sigma(t))dx \\
&-\int_\Omega \int_a^b f(x,t,\xi,u[x,g(t,\xi)])d\mu(\xi)dx,\ t \geq t_1.
\end{aligned} \tag{2.5}$$

From Green's formula, the boundary condition (1.2), we obtain

$$\begin{aligned}
\int_\Omega h_0(u)\Delta udx &= \int_{\partial\Omega} h_0(u)\frac{\partial u}{\partial \nu}dS - \int_\Omega h_0'(u)|gradu|^2 dx \\
&= -\int_\Omega h_0'(u)|gradu|^2 dx \leq 0,\ t \geq t_1,
\end{aligned} \tag{2.6}$$

$$\int_\Omega h_1(u(x,\sigma(t)))\Delta u(x,\sigma(t))dx \leq 0,\ t \geq t_1, \tag{2.7}$$

where ν is the unit exterior normal vector to $\partial\Omega$, dS is the surface element on $\partial\Omega$.
Changing order of integration and using the condition (2.2) and Jensen's inequality, we obtain

$$\begin{aligned}
&\int_\Omega \int_a^b f(x,t,\xi,u[x,g(t,\xi)])d\mu(\xi)dx \\
&= \int_a^b \int_\Omega f(x,t,\xi,u[x,g(t,\xi)])dxd\mu(\xi) \\
&\geq \int_a^b q(t,\xi) \int_\Omega F(u[x,g(t,\xi)])dxd\mu(\xi) \\
&\geq \int_a^b q(t,\xi)F((\int_\Omega dx)^{-1} \int_\Omega u[x,g(t,\xi)])dx)(\int_\Omega dx)d\mu(\xi),\ t \geq t_1. \qquad (2.8)
\end{aligned}$$

Noting that (2.1) and (2.3) and combining (2.5) – (2.8), we have

$$\frac{d^n}{dt^n}[V(t) + \int_c^d p(t,\eta)V[r(t,\eta)]d\tau(\eta)] + M\int_a^b q(t,\xi)V[g(t,\xi)]d\mu(\xi) \leq 0,\ t \geq t_1. \tag{2.9}$$

Let

$$z(t) = V(t) + \int_c^d p(t,\eta)V[r(t,\eta)]d\tau(\eta), \tag{2.10}$$

then $z(t) \geq V(t) > 0$ and form (2.9) and (2.10), we have

$$z^{(n)}(t) \leq -M\int_a^b q(t,\xi)V[g(t,\xi)]d\mu(\xi) \leq 0,\ t \geq t_1. \tag{2.11}$$

Thus, from Lemma 1.1, there exists a $t_2 \geq t_1$, such that $z'(t) > 0$ and $z^{(n-1)}(t) > 0,\ t \geq t_2$.

Form (2.10), we have

$$\begin{aligned}
V(t) &= z(t) - \int_c^d p(t,\eta)V[r(t,\eta)]d\tau(\eta) \\
&\geq z(t) - \int_c^d p(t,\eta)z[r(t,\eta)]d\tau(\eta) \\
&\geq z(t) - \int_c^d p(t,\eta)z(t)d\tau(\eta) \\
&= (1-P(t))z(t) \\
&\geq \lambda z(t),\ t \geq t_2. \qquad (2.12)
\end{aligned}$$

Combining (2.11) and (2.12) yields

$$z^{(n)}(t) \leq -\lambda MQ(t)z[g(t,a)],\ t \geq t_2. \tag{2.13}$$

Let

$$W(t) = \rho(t)\frac{z^{(n-1)}(t)}{z[\lambda g(t,a)]},\ t \geq t_2, \tag{2.14}$$

then $W(t) > 0,\ t \geq t_2$. Because $z(t)$ is increasing, $g(t,\xi)$ is nondecreasing with respect to t and ξ, there exists a $t_3 \geq t_2$, such that $z[g(t,a)] > z[\lambda g(t,a)] > 0,\ t \geq t_3$. Because $g(t,a) \leq t$ and $\frac{d}{dt}g(t,a) = g'(t,a) > 0$, from Lemma 1.2, there exists a $N > 0$ and $t_4 \geq t_3$, such that

$$z'[\lambda g(t,a)] \geq Ng^{n-2}(t,a)z^{(n-1)}[g(t,a)] \geq Ng^{n-2}(t,a)z^{(n-1)}(t),\ t \geq t_4. \tag{2.15}$$

Thus, from (2.13) – (2.15), we have

$$\begin{aligned} W'(t) &= \rho(t)\frac{z^{(n)}(t)}{z[\lambda g(t,a)]} + \frac{\rho'(t)}{\rho(t)}W(t) - \frac{\lambda\rho(t)g'(t,a)z^{(n-1)}(t)z'[\lambda g(t,a)]}{z^2[\lambda g(t,a)]} \\ &\leq -\lambda M\rho(t)Q(t) + \frac{\rho'(t)}{\rho(t)}W(t) - \frac{\lambda Ng^{n-2}(t,a)g'(t,a)}{\rho(t)}W^2(t),\ t \geq t_4. \end{aligned} \tag{2.16}$$

Let

$$X = \frac{[\lambda Ng^{n-2}(t,a)g'(t,a)]^{\frac{1}{2}}W(t)}{[\rho(t)]^{\frac{1}{2}}},\ Y = \frac{1}{2}\frac{\rho'(t)}{\rho(t)}[\frac{\rho(t)}{\lambda Ng^{n-2}(t,a)g'(t,a)}]^{\frac{1}{2}},$$

then, from the fact that $X^2 - 2XY + Y^2 \geq 0$ for any $X,\ Y \in R$, we obtain the following inequality

$$\frac{\rho'(t)}{\rho(t)}W(t) - \frac{\lambda Ng^{n-2}(t,a)g'(t,a)}{\rho(t)}W^2(t) \leq \frac{(\rho'(t))^2}{4\lambda Ng^{n-2}(t,a)g'(t,a)\rho(t)},\ t \geq t_4. \tag{2.17}$$

Thus, form (2.16) and (2.17), we have

$$W'(t) \leq -\lambda M\rho(t)Q(t) + \frac{(\rho'(t))^2}{4\lambda Ng^{n-2}(t,a)g'(t,a)\rho(t)},\ t \geq t_4. \tag{2.18}$$

Integrating both sides of (2.18) from t_4 to $t(t > t_4)$, we have

$$W(t) \leq W(t_4) - \int_{t_4}^{t}[\lambda M\rho(s)Q(s) - \frac{(\rho'(s))^2}{4\lambda Ng^{n-2}(s,a)g'(s,a)\rho(s)}]ds.$$

In the above formula, let $t \to \infty$, combining the condition (2.4), we have $\lim_{t\to\infty} W(t) = -\infty$, this contradicts the fact that $W(t) > 0$ for $t \geq t_4$. The proof of Theorem 2.1 is complete. □

Here we consider the sets

$$D_0 = \{(t,s)|t > s \geq t_0\},\ D = \{(t,s)|t \geq s \geq t_0\}.$$

Theorem 2.2. *Assume that there exists function $\rho(t),\ \varphi(t) \in C(I,R_+),\ H(t,s) \in C(D,R),\ h(t,s) \in C(D_0,R)$, such that*

(i) $H(t,t) = 0,\ t \geq t_0,\ H(t,s) > 0,\ (t,s) \in D_0$;

(ii) $H(t,s)\varphi(s)$ *exists a continuous and nonpositive partial derivative on D_0 with respect to the variable s and satisfies the equality*

$$h(t,s) = -\frac{\partial[H(t,s)\varphi(s)]}{ds} - \frac{\rho'(s)}{\rho(s)}H(t,s)\varphi(s). \tag{2.19}$$

If

$$\limsup_{t\to\infty}[\lambda M A(t,T) - \frac{1}{4\lambda N}B(t,T)] = \infty, \tag{2.20}$$

for any $T \geq t_0$, *where*

$$A(t,T) = \frac{1}{H(t,T)}\int_T^t H(t,s)\varphi(s)\rho(s)Q(s)ds,$$

$$B(t,T) = \frac{1}{H(t,T)}\int_T^t \frac{\rho(s)h^2(t,s)}{H(t,s)\varphi(s)g^{n-2}(s,a)g'(s,a)}ds,$$

then all solutions of the boundary value problems (1.1), (1.2) *are oscillatory in* G.

Proof. Proceeding as in the proof of theorem 2.1, we already have (2.16) holds. Multiplying both sides of (2.16) by $H(t,s)\varphi(s)$, for $T \geq t_4$, integrating from T to t, we have

$$\begin{aligned}\int_T^t W'(s)H(t,s)\varphi(s)ds \leq & -\lambda M\int_T^t H(t,s)\varphi(s)\rho(s)Q(s)ds + \int_T^t \frac{\rho'(s)}{\rho(s)}W(s)H(t,s)\varphi(s)ds\\ & -\lambda N\int_T^t H(t,s)\varphi(s)\rho^{-1}(s)g^{n-2}(s,a)g'(s,a)W^2(s)ds.\end{aligned}$$

Thus

$$\begin{aligned}&\lambda M\int_T^t H(t,s)\varphi(s)\rho(s)Q(s)ds\\ \leq& H(t,T)\varphi(T)W(T) - \int_T^t\{-\frac{\partial[H(t,s)\varphi(s)]}{\partial s} - \frac{\rho'(s)}{\rho(s)}W(s)H(t,s)\varphi(s)\}W(s)ds\\ &-\lambda N\int_T^t H(t,s)\varphi(s)\rho^{-1}(s)g^{n-2}(s,a)g'(s,a)W^2(s)ds\\ \leq& H(t,T)\varphi(T)W(T) + \int_T^t |h(t,s)W(s)|ds\\ &-\lambda N\int_T^t H(t,s)\varphi(s)\rho^{-1}(s)g^{n-2}(s,a)g'(s,a)W^2(s)ds.\end{aligned} \tag{2.21}$$

Let

$$X = \frac{[\lambda N H(t,s)\varphi(s)g^{n-2}(s,a)g'(s,a)]^{\frac{1}{2}}|W(s)|}{[\rho(s)]^{\frac{1}{2}}},$$

$$Y = \frac{1}{2}|h(t,s)|[\frac{\rho(s)}{\lambda N H(t,s)\varphi(s)g^{n-2}(s,a)g'(s,a)}]^{\frac{1}{2}},$$

then, from the fact that $X^2 - 2XY + Y^2 \geq 0$ for any $X,\ Y \in R$, we obtain the following inequality

$$\begin{aligned}&|h(t,s)W(s)| - \lambda N H(t,s)\varphi(s)\rho^{-1}(s)g^{n-2}(s,a)g'(s,a)W^2(s)\\ &\leq \frac{\rho(s)h^2(t,s)}{4\lambda N H(t,s)\varphi(s)g^{n-2}(s,a)g'(s,a)}.\end{aligned} \tag{2.22}$$

Combining (2.21) and (2.22), we get

$$\lambda MA(t,T) \le \varphi(T)W(T) + \frac{1}{4\lambda N}B(t,T),\ t \ge T. \tag{2.23}$$

The above formula yields

$$\limsup_{t\to\infty}[\lambda MA(t,T) - \frac{1}{4\lambda N}B(t,T)] < \infty.$$

This contradicts (2.20). The proof of Theorem 2.2 is complete. □

Corollary 2.1. *If condition* (2.20) *of Theorem 2.2 is replaced by*

$$\limsup_{t\to\infty} A(t,t_0) = \infty$$

and

$$\limsup_{t\to\infty} B(t,t_0) < \infty,$$

then the conclusions of Theorem 2.2 remain true.

If the condition (2.20) don't hold, we have the following result.

Theorem 2.3. *Assume that the other conditions of Theorem 2.2 remain unchanged, the condition* (2.20) *of Theorem 2.2 is replaced by*

$$\inf_{s\ge t_0}\{\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\} > 0 \tag{2.24}$$

and

$$\liminf_{t\to\infty} B(t,t_0) < \infty. \tag{2.25}$$

If there exists a function $\Psi(t) \in C(I,R)$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\Psi_+^2(s)g^{n-2}(s,a)g'(s,a)}{\varphi(s)\rho(s)}ds = \infty,\ \text{for every } t > t_0 \tag{2.26}$$

and

$$\liminf_{t\to\infty}\{\lambda MA(t,T) - \frac{1}{4\lambda N}B(t,T)\} \ge \Psi(T),\ \text{for every } T \ge t_0, \tag{2.27}$$

where $\Psi_+(s) = \max\{\Psi(s),0\}$, *the definitions of* $A(t,T)$ *and* $B(t,T)$ *see (2.20), then all solutions of the boundary value problems* (1.1), (1.2) *are oscillatory in* G.

Proof. Proceeding as in the proof of theorem 2.2, for any $t \ge T \ge t_4$, , we already have (2.23) holds, then

$$\lambda MA(t,T) - \frac{1}{4\lambda N}B(t,T) \le \varphi(T)W(T),\ t \ge T. \tag{2.28}$$

From (2.27) and (2.28), we have

$$\Psi(T) \le \varphi(T)W(T),\ T \ge t_4 \tag{2.29}$$

and

$$\liminf_{t\to\infty} \lambda MA(t,t_4) \geq \Psi(t_4). \tag{2.30}$$

From (2.26) and (2.29), we obtain

$$\int_{t_4}^{\infty} \varphi(s)\rho^{-1}(s)g^{n-2}(s,a)g'(s,a)W^2(s)ds = \infty. \tag{2.31}$$

To complete the proof of this theorem, we merely need to prove that (2.31) is impossible. For this purpose, we definite

$$F(t) = \frac{1}{H(t,t_4)} \int_{t_4}^{t} |h(t,s)W(s)|ds,$$

$$G(t) = \frac{\lambda N}{H(t,t_4)} \int_{t_4}^{t} H(t,s)\varphi(s)\rho^{-1}(s)g^{n-2}(s,a)g'(s,a)W^2(s)ds.$$

From (2.21) and (2.30), we have

$$\begin{aligned}\limsup_{t\to\infty}[G(t)-F(t)] &\leq \varphi(t_4)W(t_4) - \liminf_{t\to\infty} \lambda MA(t,t_4)\\ &\leq \varphi(t_4)W(t_4) - \Psi(t_4)\\ &< \infty.\end{aligned} \tag{2.32}$$

From (2.24) and (2.31), we obtain

$$\lim_{t\to\infty} G(t) = \infty. \tag{2.33}$$

Now, let we consider a sequence $\{t_k\}_{k=1}^{\infty} \subset I$ with $\lim_{k\to\infty} t_k = \infty$. From (2.32), there exists a constant C such that

$$G(t_k) - F(t_k) \leq C,\ \ k = 1, 2, \cdots . \tag{2.34}$$

From (2.33), we have

$$\lim_{k\to\infty} G(t_k) = \infty. \tag{2.35}$$

Combining (2.34) and (2.35), we get

$$\lim_{k\to\infty} F(t_k) = \infty, \tag{2.36}$$

and

$$\frac{F(t_k)}{G(t_k)} - 1 \geq -\frac{C}{G(t_k)} > -\frac{1}{2},$$

namely,

$$\frac{F(t_k)}{G(t_k)} > \frac{1}{2},\ \text{for sufficiently large } k.$$

From the above formula and (2.36), we have

$$\lim_{k\to\infty} \frac{F^2(t_k)}{G(t_k)} = \infty. \tag{2.37}$$

On the other hand, by using the Schwarz inequality, we obtain

$$\begin{aligned} F(t_k) \leq &\{\frac{\lambda N}{H(t_k,t_4)}\int_{t_4}^{t_k} H(t_k,s)\varphi(s)\rho^{-1}(s)g^{n-2}(s,a)g'(s,a)W^2(s)ds\}^{\frac{1}{2}} \\ &\times\{\frac{1}{\lambda N H(t_k,t_4)}\int_{t_4}^{t_k}\frac{\rho(s)h^2(t_k,s)}{H(t_k,s)\varphi(s)g^{n-2}(s,a)g'(s,a)}ds\}^{\frac{1}{2}}. \end{aligned}$$

Thus, we have

$$\frac{F^2(t_k)}{G(t_k)} \leq \frac{1}{\lambda N}B(t_k,t_4), \text{ for sufficiently large } k.$$

Noting that (2.37), we obtain

$$\lim_{k\to\infty} B(t_k,t_4) = \infty. \tag{2.38}$$

Because the sequence $\{t_k\}_{k=1}^{\infty}$ is arbitrary, (2.38) contradicts (2.25). Thus, (2.31) doesn't hold. The proof of Theorem 2.3 is complete. □

Remark 2.1. The results of this paper extend and improve the corresponding oscillatory theorems of literature [11].

Remark 2.2. Using our ideas in this paper, we can consider the other boundary conditions. For example, consider the following Robin boundary value condition

$$\frac{\partial u}{\partial N} + \beta(x)u = 0,\ (t,x) \in R_+ \times \partial\Omega, \tag{2.39}$$

where $\beta(x) \in C(\partial\Omega,(0,\infty))$. It is not difficult to obtain some oscillation criteria of the boundary value problems (1.1), (2.39). Due to limited space, their statements are omitted here.

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