Journal of Nonlinear Modeling and Analysis
Volume 1, Number 2, June 2019, 207–220 Website:http://jnma.ca
DOI:10.12150/jnma.2019.207

On the Integrability and Equivalence of the Abel Equation and Some Polynomial Equations*

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Abstract In this paper, first of all we give the necessary and sufficient conditions of the center of a class of planar quintic differential systems by using reflecting function method, and provide a simple proof of this results. Secondly, We use the reflecting integral to research the equivalence of the Abel equation and some complicated equations and derive their center conditions and discuss their integrability.

Keywords Center condition, integrability, reflecting integral, equivalence.

MSC(2010) 34C07, 34C05, 34C25, 37G15.

1. Introduction

In this paper, we will consider the Abel equations of the form

$$\frac{dr}{d\theta} = A(\theta)r^2 + B(\theta)r^3, \tag{1.1}$$

where $A(\theta)$, $B(\theta)$ are continuous functions. The main reason why we are interested in this Abel equations is that they are closely related to planar vector fields. There are many classes of planar systems which are in some sense equivalent to some Abel equations [1–4, 6, 7, 14, 15]. The first class is planar polynomial systems of the form $x' = -y + p$, $y' = x + q$ with homogeneous polynomials p and q of degree k. The second class is the Liénard systems : $x' = y$, $y' = -f(x)y - g(x)$, they can be transformed to the Abel (1.1) [15]. The third class is the system

$$\begin{cases} x' = -y + x(P_n(x,y) + P_{2n}(x,y)), \\ y' = x + y(P_n(x,y) + P_{2n}(x,y)), \end{cases} \tag{1.2}$$

where $P_k(x,y) = \sum_{i+j=k} p_{ij}x^iy^j$, $p_{i,j}$ $(i,j = 0,1,2,...,k,\ k = n, 2n)$ are real constants.

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*The authors were supported by the National Natural Science Foundation of China (61773017, 11571301), and the National Natural Science Foundation of Province Jiangsu (BK20161327).

In polar coordinates, the system (1.2) becomes

$$\frac{d\rho}{d\theta} = (P_n(\cos\theta, \sin\theta) + P_{2n}(\cos\theta, \sin\theta)\rho^n)\rho^{n+1}. \tag{1.3}$$

Taking $r = \rho^n$, then (1.3) becomes (1.1) with $A(\theta) = nP_n$, $B(\theta) = nP_{2n}$. The origin is a center for the two-dimensional system (1.2) if and only if all solutions of the Abel equation (1.1) starting near the origin are periodic with period 2π , *i.e.*, all the solutions nearby are closed: $r(2\pi) = r(0)$. In this case, we say that $r = 0$ is a center of the Abel equation.

The Abel equations have been investigated over the years. In the papers [1–4,6,7,14,15] and others, the authors Alwash and Lloyd presented some center conditions for the Abel equations and give the composition conditions [2,3] under which the Able equation has a center. Yomdin [6] and Yang [15] give an asymptotic expansion of the solutions of Abel equations and some center conditions.

In this paper, in the first section, we use reflecting function method [9,17,18] to derive the center conditions for a class of planar quintic differential systems and provide a simple proof of this results. In the second section, we give the integrability conditions of some polynomial differential equations by using its reflecting integrals [17], and establish the equivalence between the polynomial equations and some complicated equations and judge when do these complicated equations have a center at the origin.

Now, I briefly introduce the concept of the reflecting function and reflecting integral which will be used throughout the rest of this article.

Consider differential system

$$x' = X(t, x), \ (t \in I \subset R, \, x \in D \subset R^n, \, 0 \in I) \tag{1.4}$$

which has a continuously differentiable right-hand side and general solution $\varphi(t; t_0, x_0)$.

Definition 1.1. [9] For system (1.4), $F(t, x) := \varphi(-t, t, x)$ is called its **Reflecting function**.

By this, for any solution $x(t)$ of (1.4), we have $F(t, x(t)) = x(-t), F(0, x) = x$ and $F(t, x)$ is a reflecting function of system (1.4), if and only if, it is a solution of the Cauchy problem

$$F_t + F_x X(t, x) + X(-t, F) = 0, F(0, x) = x. \tag{1.5}$$

By [9,18], if system (1.4) is 2ω-periodic with respect to t, and $F(t, x)$ is its reflecting function, then $T(x) := F(-\omega, x)$ is the Poincaré mapping of (1.4) over the period $[-\omega, \omega]$, and the solution $x = \varphi(t; -\omega, x_0)$ of (1.4) defined on $[-\omega, \omega]$ is 2ω-periodic if and only if x_0 is a fixed point of $T(x)$. Thus, we can use the method of reflecting function to study the existence and stability of the periodic solutions of the differential systems (1.4) [5,9–13,17,18].

Definition 1.2. [9] If the reflecting functions of two differential systems coincide in their common domain, then these systems are said to be **Equivalent**.

Definition 1.3. [17] If $\Delta(t, x)$ is a unequal identically to zero solution of the partial differential system

$$\Delta_t(t, x) + \Delta_x(t, x)X(t, x) - X_x(t, x)\Delta(t, x) = 0, \tag{1.6}$$

then $\Delta(t, x)$ is called a **Reflecting integral** of (1.4).

Lemma 1.1 ([9,10,17]). *If $\Delta_i(t,x)\,(i=1,2,...,m)$ are the reflecting integrals of* (1.4)*, then the system* (1.4) *is equivalent to the system*

$$x' = X(t,x) + \sum_{i=1}^{m} \alpha_i(t, u_1(t,x), u_2(t,x), ..., u_n(t,x))\Delta_i(t,x),$$

where $\alpha_i(t,u_1,u_2,...,u_n)\,(i=1,2,..,m)$ are arbitrary odd continuously differentiable scalar functions with respect to t, $u_i(t,x)(i=1,2,...,n)$ are the independent first integrals of (1.4)*.*

In particular, if $n=1$ in (1.4) we get the following result.

Lemma 1.2 ([17]). *For one-dimensional differential equation* (1.4) *(in which $n=1$), if $\Delta(t,x)$ is its reflecting integral, then Δ^{-1} is the integral factor of* (1.4) *and one equation is equivalent to equation* (1.4)*, if and only if, it can be expressed as*

$$x' = X(t,x) + \alpha(t,u)\Delta(t,x), \tag{1.7}$$

where $\alpha(t,u)$ is a continuously differentiable odd function with respect to t, u is the first integral of (1.4)*.*

Besides, if the equations (1.4) *and* (1.7) *are $2\pi-$ periodic with respect to t, then the qualitative behavior of the 2π- periodic solutions of these equivalent systems are the same.*

By Lemma1.2, to find the reflecting integral is very important for discussing the equivalence and integrability of some differential equations. In general, to find out the reflecting integral Δ from (1.6) is very difficult. Belsky [5] and Musafirov [13] and Zhou [17] have found some special reflecting integrals for some differential equations and using them to discuss the qualitative behavior of the time-varying differential systems. In this paper, we will give the sufficient condition for equations (1.1) and (1.3) have some polynomial reflecting integrals, and use them to discuss the integrability of these polynomial equations and give the expression of its first integrals. We establish the equivalence between the polynomial equations and some complicated systems and give their center conditions.

In the following, all the differential systems under discussing have continuously differentiable right-hand sides, and have a unique solution for their initial value problems.

2. Center condition for a class of planar quintic system

In [2,4], Alwash and Lloyd give the following conclusion.

Lemma 2.1 ([2,4]). *If there exists a differentiable function u of period 2π such that*

$$A(\theta) = u'(\theta)A_1(u(\theta)),\ B(\theta) = u'(\theta)B_1(u(\theta))$$

for some continuous functions A_1 and B_1, then the Abel differential equation (1.1) *has a center at the origin.*

The condition in Lemma 2.1 is called the **Composition Condition**. This is a sufficient but not a necessary condition for the origin to be a center [2].

Consider the quintic differential system

$$\begin{cases} x' = -y + x(P_2(x,y) + P_4(x,y)), \\ y' = x + y(P_2(x,y) + P_4(x,y)), \end{cases} \tag{2.1}$$

where $P_n(x,y) = \sum_{i+j=n} p_{ij} x^i y^j$, $p_{i,j}$ $(i,j = 0,1,2,...,n,\ n = 2,4)$ are real constants.

Although there are several authors [8, 14] have solved this problem by using computer and other traditional methods. But, in this paper, we also show how to use the reflecting function method [9,17] (a new method) to derive the center conditions and give the simple proof of the obtained results. Moreover, this result will help us to study the integrability of the corresponding equivalent periodic equation later.

In this section, we denote:

$$\bar{A} = \int_0^\theta A d\theta,\ A_e = \frac{1}{2}(A(\theta) + A(-\theta)),\ A_o = \frac{1}{2}(A(\theta) - A(-\theta)), etc.$$

Theorem 2.1. *The origin is a center for* (2.1)*, if and only if, the following conditions are satisfied*

$$p_{20} + p_{02} = 0; \tag{2.2}$$

$$p_{22} + 3(p_{40} + p_{04}) = 0; \tag{2.3}$$

$$p_{11}(p_{04} - p_{40}) + p_{20}(p_{31} + p_{13}) = 0; \tag{2.4}$$

$$(p_{11}^2 - 4p_{20}^2)(p_{40} + p_{04}) - p_{11}p_{20}(p_{31} - p_{13}) = 0. \tag{2.5}$$

Proof. Taking $x = \rho\cos\theta$, $y = \rho\sin\theta$, system (2.1) becomes

$$\begin{cases} \frac{d\rho}{dt} = \rho^3(P_2 + P_4\rho^2), \\ \frac{d\theta}{dt} = 1, \end{cases}$$

where

$$P_2 = p_{20}\cos^2\theta + p_{11}\cos\theta\sin\theta + p_{02}\sin^2\theta,$$

$$P_4 = p_{40}\cos^4\theta + p_{31}\cos^3\theta\sin\theta + p_{22}\cos^2\theta\sin^2\theta + p_{13}\cos\theta\sin^3\theta + p_{04}\sin^4\theta.$$

By [1–4,15], we know that the origin $(0,0)$ of (2.1) is a center, if and only if, every solution in a neighborhood of $\rho = 0$ is a $2\pi-$ periodic solution for the differential equation

$$\frac{d\rho}{d\theta} = \rho^3(P_2 + P_4\rho^2). \tag{2.6}$$

If we put $r = \rho^2$, then equation (2.6) will be transformed to

$$\frac{dr}{d\theta} = A(\theta)r^2 + B(\theta)r^3, \tag{2.7}$$

where

$$A(\theta) = 2P_2(\cos\theta, \sin\theta),\ B(\theta) = 2P_4(\cos\theta, \sin\theta).$$

Let $F(\theta, r)$ be the reflecting function of (2.7) with $F(0,r) = r$. We write

$$F(\theta, r) = \sum_{n=1}^{\infty} a_n(\theta) r^n, \tag{2.8}$$

where $a_1(0) = 1$ and $a_n(0) = 0$ for $n > 1$. The origin is a center if and only if $F(\theta + 2\pi, r) = F(\theta, r)$, *i.e.*, $a_1(2\pi) = 1$, $a_n(2\pi) = 0\,(n = 2, 3, 4, ...)$ [9, 17, 18]. Substituting (2.8) into (1.5) with $X = Ar^2 + Br^3$ we get

$$\sum_{n=1}^{\infty} a_n'(\theta) r^n + (\sum_{n=1}^{\infty} n a_n(\theta) r^{n-1})(Ar^2 + Br^3) \\ + A(-\theta)(\sum_{n=1}^{\infty} a_n(\theta) r^n)^2 + B(-\theta)(\sum_{n=1}^{\infty} a_n(\theta) r^n)^3 = 0.$$

Equating the corresponding coefficients of r^n yields

$$a_1' = 0, a_1(0) = 0;$$

$$a_n' + (n-1)a_{n-1}A + (n-2)a_{n-2}B + A(-\theta)\sum_{i+j=n} a_i a_j + B(-\theta)\sum_{i+j+k=n} a_i a_j a_k = 0,$$

$$a_n(0) = 0,\ n = 2, 3, 4, ...$$

Solving these equations recursively gives

$$a_1 = 1,$$

$$a_2' = -2A_e,\ a_2(0) = 0; \tag{2.9}$$

$$a_3' = -4a_2A_e - 2B_e, a_3(0) = 0; \tag{2.10}$$

$$a_4' = -3a_3A - 2a_2B - 3a_2B(-\theta) - A(-\theta)(a_2^2 + 2a_3),\ a_4(0) = 0; \tag{2.11}$$

$$a_5' = -4a_4A - 3a_3B - 2(a_2a_3 + a_4)A(-\theta) - 3(a_2^2 + a_3)B(-\theta),\ a_5(0) = 0. \tag{2.12}$$

Solving (2.9) we get

$$a_2(\theta) = -2\int_0^{\theta} A_e d\theta = -2\bar{A}_e$$

and

$$a_2(2\pi) = -4(p_{20} + p_{02})\pi,$$

it implies that $a_2(2\pi) = 0$, i.e., $p_{20} + p_{02} = 0$, thus the first condition (2.2) is valid. Using this and (2.10), we have

$$a_3(\theta) = a_2^2 - 2\bar{B}_e$$

and

$$a_3(2\pi) = -2\int_0^{2\pi} B_e d\theta = -\pi(3(p_{40} + p_{04}) + p_{22}),$$

from $a_3(2\pi) = 0$ implies the second necessary condition (2.3) holds. Using (2.2) and (2.3) we get

$$a_4(\theta) = a_2^3 + 10\bar{A}_e\bar{B}_e + 2\overline{(A_o\bar{B}_e - B_o\bar{A}_e)}$$

and

$$a_4(2\pi) = 2\int_0^{2\pi} (A_o\bar{B}_e - B_o\bar{A}_e) d\theta = \pi(p_{11}(p_{40} - p_{04}) - p_{20}(p_{31} + p_{13})),$$

so, by $a_4(2\pi) = 0$ we know the third necessary condition (2.4) is valid. Applying the above relations and (2.12), we have

$$a_5(\theta) = a_2^4 + 6\bar{B}_e^2 - 36\bar{A}_e^2\bar{B}_e - 12\bar{A}_e\overline{A_o\bar{B}_e - \bar{A}_eB_o} + 4\overline{A_e\bar{A}_e\bar{B}_e - A_o\overline{(A_o\bar{B}_e - \bar{A}_eB_o)}},$$

$$a_5(2\pi) = 4\int_0^{2\pi}(A_e\bar{A}_e\bar{B}_e - A_o\overline{(A_o\bar{B}_e - \bar{A}_eB_o)})d\theta =$$

$$= \frac{\pi}{4}((4p_{20}^2 - p_{11}^2)(p_{40} + p_{04}) + p_{11}p_{20}(p_{31} - p_{13})),$$

so, by $a_5(2\pi) = 0$ we get

$$(p_{11}^2 - 4p_{20}^2)(p_{40} + p_{04}) - p_{11}p_{20}(p_{31} - p_{13}) = 0.$$

Thus the necessity of the present theorem is established.

Now, we prove that these conditions are also sufficient.

Case 1. If $p_{13} + p_{31} = 0$. From (2.4) follows $p_{11} = 0$ or $p_{04} - p_{40} = 0$.

1^0. If $p_{11} = 0$, from (2.5), we get $p_{20} = 0$ or $p_{40} + p_{04} = 0$.

If $p_{20} = 0$, by relation (2.2), we have $P_2 \equiv 0$. On the other hand, when relation (2.3) is held, $\int_0^{2\pi} P_4 d\theta = 0$. Thus solving the first order equation (2.7), we get $r(\theta)$ is a 2π-periodic, so $r = 0$ is a center.

If $p_{20} \neq 0$, $p_{40} + p_{04} = 0$, then $P_2 = p_{20}\cos 2\theta$,

$$P_4 = \frac{P_2}{p_{20}}(p_{40} + \frac{p_{31}}{p_{20}}\overline{P}_2).$$

By Lemma 2.1, the origin of (2.7) is a center.

2^0. If $p_{11} \neq 0$, $p_{04} - p_{40} = 0$.

If $p_{20} \neq 0$, then

$$P_4 = \frac{p_{40}}{p_{20}p_{11}}(p_{11} - 4\bar{P}_2)P_2.$$

If $p_{20} = 0$, then

$$P_4 = \frac{p_{31}}{p_{11}^2}(p_{11} - 4\bar{P}_2)P_2.$$

As P_2 is a 2π-periodic function, by Lemma 2.1, the origin of (2.7) is a center.

Case 2. If $p_{13} + p_{31} \neq 0$, then from (2.4) we get

$$p_{20} = \frac{p_{11}(p_{40} - p_{04})}{p_{31} + p_{13}}. \tag{2.13}$$

Substituting (2.13) into (2.5) we obtain

$$\frac{p_{11}^2}{(p_{31} + p_{13})^2}[(p_{40} + p_{04})((p_{31} + p_{13})^2 - 4(p_{40} - p_{04})^2) - (p_{40} - p_{04})(p_{31}^2 - p_{13}^2)] = 0. \tag{2.14}$$

1^0. If $p_{11} \neq 0$, then from (2.14) follows that

$$(p_{31} + p_{13})(p_{04}p_{31} + p_{40}p_{13}) = 2(p_{04} - p_{40})^2(p_{40} + p_{04}). \tag{2.15}$$

Using relations (2.15) and (2.2) and (2.3) we get:

If $p_{20} \neq 0$, then

$$P_4 = \frac{1}{p_{20}p_{11}}(p_{11}p_{40} - 2(p_{40} + p_{04})\bar{P}_2)P_2.$$

If $p_{20} = 0$, then from (2.4) and (2.5) follows $p_{40} = p_{04} = p_{22} = 0$ and

$$P_4 = \frac{1}{p_{11}^2}(p_{11}p_{31} + 2(p_{13} - p_{31})\bar{P}_2)P_2.$$

By Lemma 2.1, the origin of (2.7) is a center.

2^0. If $p_{11} = 0$, then from (2.14) and (2.2) follows $p_{20} = p_{02} = 0$, so $P_2 \equiv 0$. Thus, by (2.3) and (2.7) we know $r = 0$ is a center.

In summary, the proof of the present theorem is finished. □

3. Equivalence and integrability of the Abel equation and some polynomial equations

Consider the Abel equation

$$\frac{dr}{d\theta} = A(\theta)r^2 + B(\theta)r^3 = R(\theta, r), \tag{3.1}$$

where $A(\theta)$, $B(\theta)$ are continuous functions, $\theta \in D \in R$.

Obviously, if $B = kA$ (k is a constant), or $AB \equiv 0$, then this Abel equation is integrable. So, in the following, we only discuss (3.1) in the other cases.

By [5,17], if the reflecting integral of (3.1) is a polynomial function on r, then it must be a cubic function. If $\Delta(\theta, r)$ and $\tilde{\Delta}(\theta, r)$ are the reflecting integrals of (3.1), then $\tilde{\Delta}(\theta, r) = \Delta(\theta, r)\phi(u)$, where u is the first integral of (3.1), this means that if we can find out a reflecting integral of (3.1), at the same time, we know its infinite reflecting integrals. So, in the following we only discuss when does (3.1) have a cubic reflecting integral ? By Lemma 1.2, we can use the reflecting integral for discussing the equivalence and integrability of (3.1) and some differential equations and derive their center conditions.

Let

$$v_0 = 4A\overline{B\bar{A}} - 3B\bar{B} - B\bar{A}^2;\ v_1 = 2A\bar{B} - B\bar{A};\ v_2 = -B;\ v_3 = A, \tag{3.2}$$

where $\bar{A}(\theta) = \int A(\theta)d\theta$, $\bar{B} = \int B(\theta)d\theta$, *etc.*

Theorem 3.1. *Suppose that the functions v_0, v_1, v_2, v_3 are linear dependent, i.e., there are $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \neq 0$ such that*

$$\lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0,\ \theta \in D. \tag{3.3}$$

Then

$$\Delta = a_0(\theta) + a_1(\theta)r + a_2(\theta)r^2 + a_3(\theta)r^3$$

is a reflecting integral of the Abel equation (3.1) *and $u = \int \frac{1}{\Delta}dr$ is the first integral of* (3.1) *and which is equivalent to equation*

$$\frac{dr}{d\theta} = A(\theta)r^2 + B(\theta)r^3 + \alpha(\theta, u)\Delta(\theta, r), \tag{3.4}$$

where $\alpha(\theta, u)$ is an arbitrary continuously differentiable odd function with respect to θ,

$$a_0 = \lambda_0;\ a_1 = 2\lambda_0\bar{A} + \lambda_1;\ a_2 = \lambda_0(\bar{A}^2 + 3\bar{B}) + \lambda_1\bar{A} + \lambda_2;\ a_3 = 4\lambda_0\overline{B\bar{A}} + 2\lambda_1\bar{B} + \lambda_3. \tag{3.5}$$

Besides, if (3.1) *and* (3.4) *are 2π-periodic equations, then the qualitative behavior of the 2π-periodic solutions of their are the same.*

Proof. By Definition 1.3, the cubic function

$$\Delta = a_0(\theta) + a_1(\theta)r + a_2(\theta)r^2 + a_3(\theta)r^3$$

is the reflecting integral of (3.1), if and only if, it is a solution of

$$\Delta_\theta(\theta, r) + \Delta_r(\theta, r)R(\theta, r) - R_r(\theta, r)\Delta(\theta, r) = 0, \tag{3.6}$$

i.e.,

$$\begin{aligned} a_0' + a_1'r + a_2'r^2 + a_3'r^3 + (a_1 + 2a_2r + 3a_3r^2)(Ar^2 + Br^3) \\ - (2Ar + 3Br^2)(a_0 + a_1r + a_2r^2 + a_3r^3) = 0. \end{aligned}$$

Equating the coefficients of r yield

$$a_0' = 0;\ a_1' = 2Aa_0;\ a_2' = a_1A + 3a_0B;\ a_3' = 2a_1B;\ a_3A = a_2B.$$

Solving these equations we get (3.5) and (3.3) and (3.2). This shows that the function $\Delta = a_0(\theta) + a_1(\theta)r + a_2(\theta)r^2 + a_3(\theta)r^3$ is the reflecting integral of (3.1). By Lemma 1.2, the equation (3.1) is equivalent to (3.4) and if they are 2π-periodic systems, then the qualitative behavior of their 2π-periodic solutions are the same, Δ^{-1} is the integral factor of equation (3.1) and its first integral is

$$u = \int_{(0,0)}^{(\theta,r)} \frac{1}{\Delta(\theta, r)}dr - \frac{R(\theta, r)}{\Delta(\theta, r)}d\theta,$$

since $\frac{R(\theta,r}{\Delta(\theta,r)}|_{r=0} = 0$, so, $u = \int \frac{1}{\Delta(\theta,r)}dr$. □

Theorem 3.2. *If $B = kA\bar{A}$, k is a constant, then the Abel equation* (3.1) *has reflecting integral*

$$\Delta = r + \bar{A}r^2 + 2\bar{B}r^3.$$

The first integral of (3.1) *is as follows:*

1^0. *If $k = \frac{1}{4}$,*

$$u = \ln\frac{2r}{2 + \bar{A}r} + \frac{2}{2 + \bar{A}r};$$

2^0. *If $k > \frac{1}{4}$,*

$$u = \frac{1}{2}\ln\frac{r^2}{|1 + \bar{A}r + 2\bar{B}r^2|} - \frac{1}{\sqrt{4k-1}}\arctan\frac{2k\bar{A}r + 1}{\sqrt{4k-1}};$$

3^0. *If $k < \frac{1}{4}$,*

$$u = \frac{1}{2}\ln\frac{r^2}{|1 + \bar{A}r + 2\bar{B}r^2|} - \frac{1}{2\sqrt{1-4k}}\ln|\frac{2k\bar{A}r + 1 - \sqrt{1-4k}}{2k\bar{A}r + 1 + \sqrt{1-4k}}|.$$

Proof. If $B = kA\bar{A}$, then $v_1 = 2\bar{B}A - B\bar{A} = k(A\bar{A}^2 - A\bar{A}^2) = 0$ and the functions v_0, v_1, v_2, v_3 are linear dependent. Taking $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = (0, 1, 0, 0)$ we have

$$\lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0.$$

Therefore, by (3.5) we get

$$a_0 = 0,\ a_1 = 1,\ a_2 = \bar{A},\ a_3 = 2\bar{B}$$

and

$$\Delta = r + \bar{A}r^2 + 2\bar{B}r^3$$

is the reflecting integral of (3.1).

By Lemma 1.2, Δ^{-1} is the integral factor of (3.1), then its first integral is

$$u = \int \frac{1}{\Delta} dr + \psi(\theta) = \int \frac{1}{r + \bar{A}r^2 + 2\bar{B}r^3} dr + \psi(\theta)$$

and

$$\psi'(\theta) = -\frac{R}{\Delta} - \frac{\partial}{\partial\theta}(\int \frac{1}{\Delta} dr) = -\frac{R}{\Delta}|_{r=0} - \frac{\partial}{\partial\theta}(\int \frac{1}{\Delta} dr)|_{r=0}.$$

On the other hand,

$$\int \frac{1}{\Delta} dr = \int \frac{1}{r(1 + \bar{A}r + 2\bar{B}r^2)} dr = \int (\frac{1}{r} - \frac{\bar{A} + 2\bar{B}r}{1 + \bar{A}r + 2\bar{B}r^2}) dr = \ln|r| - \int (\bar{A} + 2\bar{B}r)$$
$$(1 + \sum_{i=1}^{\infty} (-1)^i (\bar{A}r + 2\bar{B}r^2)^i) dr = \ln|r| - \bar{A}r + (A\bar{A} - 2\bar{B})r^2 + \ldots.$$

From this relation we get

$$\frac{\partial}{\partial\theta}(\int \frac{1}{\Delta} dr)|_{r=0} = 0.$$

Thus $\psi'(\theta) = 0$ and $u = \int \frac{1}{r + \bar{A}r^2 + 2\bar{B}r^3} dr$, integrating this indefinite integral we obtain the expression of u as the above. □

Remark 3.1. In particular, in the case of $B = \frac{2}{9}\bar{A}$, we can easily to get the first integral of (3.1). As $B = \frac{2}{9}A\bar{A}$, $v_0 = 0$, $v_1 = 0$ and the functions v_0, v_1, v_2, v_3 are linear dependent. Taking $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = (0, 1, 0, 0)$ or $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = (1, 0, 0, 0)$, the relation (3.3) is held. Using (3.5) we obtain two reflecting integrals:

$$\Delta = r + \bar{A}r^2 + 2\bar{B}r^3 = r(1 + \frac{2}{3}\bar{A}r)(1 + \frac{1}{3}\bar{A}r)$$

and

$$\tilde{\Delta} = 1 + 2\bar{A}r + (\bar{A}^2 + 3\bar{B})r^2 + 4\overline{B\bar{A}}r^3 = (1 + \frac{2}{3}\bar{A}r)^3.$$

By [17],

$$u = \frac{\Delta}{\tilde{\Delta}} = \frac{3r(3 + \bar{A}r)}{(3 + 2\bar{A}r)^2}$$

is the first integral of (3.1) and which is equivalent to equation (3.4).

Corollary 3.1. *If*

$$p_{20}+p_{02}=0,\ p_{20}p_{10}^2+p_{11}p_{10}p_{01}+p_{02}p_{01}^2=0,$$

then system

$$\begin{cases}\frac{dx}{dt}=-y+x(P_1(x,y)+P_2(x,y))+x\alpha(\theta,u)(1+p_{10}y-p_{01}x+2p_{20}xy+p_{11}y^2),\\ \frac{dy}{dt}=x+y(P_1(x,y)+P_2(x,y))+y\alpha(\theta,u)(1+p_{10}y-p_{01}x+2p_{20}xy+p_{11}y^2)\end{cases}\tag{3.7}$$

has a center at (0,0), where

$$P_1(x,y)=p_{10}x+p_{01}y,\ P_2(x,y)=p_{20}x^2+p_{11}xy+p_{02}y^2,$$

$\alpha(\theta,u)$ *is an arbitrary continuously differentiable* 2π*-periodic odd function with respect to* θ*,*

$$\theta=\arctan\frac{y}{x},\ u=\int(r+\bar{P}_1r^2+2\bar{P}_2r^3)^{-1}dr,\ r=\sqrt{x^2+y^2},$$

$$P_1=P_1(\cos\theta,\sin\theta),\ P_2=P_2(\cos\theta,\sin\theta),\ \bar{P}_1=\int_0^\theta P_1d\theta,\ \bar{P}_2=\int_0^\theta P_2d\theta.$$

Proof. By the assumption and [1], we know the system

$$\begin{cases}\frac{dx}{dt}=-y+x(P_1(x,y)+P_2(x,y)),\\ \frac{dy}{dt}=x+y(P_1(x,y)+P_2(x,y))\end{cases}\tag{3.8}$$

has a center at (0,0) and $P_2=kP_1\bar{P}_1$, k is a constant. By Theorem 3.2, the equation

$$\frac{dr}{d\theta}=r^2P_1+P_2r^3\tag{3.9}$$

is equivalent to equation

$$\frac{dr}{d\theta}=r^2P_1+P_2r^3+\alpha(\theta,u)\Delta(\theta,u),\tag{3.10}$$

where $\Delta=r+\bar{P}_1r^2+2\bar{P}_2r^3$, $u=\int\frac{1}{\Delta}dr$.

Taking $x=r\cos\theta$, $y=r\sin\theta$, equation (3.10) is transformed to (3.7), by Lemma 1.2, system (3.7) has a center at (0,0) as well. □

Example 3.1. If the conditions of Corollary 3.1 are satisfied, and $p_{10}p_{01}=9$, $p_{20}=-2$, then $P_2=\frac{2}{9}P_1\bar{P}_1$ and

$$\Delta=r(3+2\bar{P}_1r)(3+\bar{P}_1r),\ u=\frac{r(3+\bar{P}_1r)}{(3+2\bar{P}_1r)^2}.$$

If we take $\alpha=\lambda u\sin\theta$, λ is a constant, then (3.10) becomes

$$\frac{dr}{d\theta}=P_1r^2+P_2r^3+\sin\theta\frac{r^2(3+\bar{P}_1r)^2}{3+2\bar{P}_1r}.$$

By this we get its equivalent system

$$\begin{cases} \frac{dx}{dt} = (-y + x(P_1(x,y) + P_2(x,y))(1 + \frac{2}{3}(p_{10}y - p_{01}x)) + \frac{\lambda}{3}xy(3 + p_{10}y - p_{01}x)^2, \\ \frac{dy}{dt} = (x + y(P_1(x,y) + P_2(x,y))(1 + \frac{2}{3}(p_{10}y - p_{01}x)) + \frac{\lambda}{3}y^2(3 + p_{10}y - p_{01}x)^2 \end{cases}$$

has a center at (0,0) and has a first integral :

$$H = ue^{\lambda\cos\theta} = \frac{(3 + p_{10}y - p_{01}x)\sqrt{x^2+y^2}}{(3 + 2(p_{10}y - p_{01}x))^2} e^{\frac{\lambda x}{\sqrt{x^2+y^2}}}.$$

This example shows that, if one system has a center at origin, by the equivalence we will know its many equivalent systems have a center at origin, too.

Theorem 3.3. *If* $B = (k_0 + k_1\bar{A})A$, k_0, k_1 *are nonzero constants, then the Abel equation* (3.1) *has a reflecting integral*

$$\Delta = kr + (1 + k\bar{A})r^2 + (k_0 + 2k\bar{B})r^3,$$

where $k = \frac{k_1}{k_0}$. *The first integral of* (3.1) *is as follows:*

1^0. *If* $k_1 = \frac{1}{4}$,

$$u = \frac{1}{2k}\ln\frac{4k_0r^2}{(1 + 2k_0(1 + k\bar{A})r)^2} + \frac{1}{k(1 + 2k_0(1 + k\bar{A})r)};$$

2^0. *If* $k_1 > \frac{1}{4}$,

$$u = \frac{1}{2k}\ln\frac{r^2}{|k + (1 + k\bar{A})r + (k_0 + 2k\bar{B})r^2|} - \frac{1}{k\sqrt{4k_1 - 1}} arctan\frac{1 + 2k_0(1 + k\bar{A})r}{\sqrt{4k_1 - 1}};$$

3^0. *If* $k_1 < \frac{1}{4}$,

$$\begin{aligned} u =& \frac{1}{2k}\ln\frac{r^2}{|k + (1 + k\bar{A})r + (k_0 + 2k\bar{B})r^2|} \\ &- \frac{1}{2k\sqrt{1 - 4k_1}}\ln|\frac{1 + 2k_0(1 + k\bar{A})r - \sqrt{1 - 4k_1}}{1 + 2k_0(1 + k\bar{A})r + \sqrt{1 - 4k_1}}|. \end{aligned}$$

4^0. *If* $k_1 = \frac{2}{9}$,

$$u = \frac{r(2 + 3(k_0 + \frac{2}{9}\bar{A})r)}{(1 + 3(k_0 + \frac{2}{9}\bar{A})r)^2}.$$

Furthermore, the Abel equation (3.1) *is equivalent to equation* (3.4).

Proof. As $B = (k_0 + k_1\bar{A})A$, using (3.2) we get $v_2 = -k_0v_3 - kv_1$, *i.e.*, the functions v_0, v_1, v_2, v_3 are linear dependent. Taking $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = (0, k, 1, k_0)$ we get

$$\lambda_0v_0 + \lambda_1v_1 + \lambda_2v_2 + \lambda_3v_3 = 0.$$

By Theorem 3.1 and (3.5) we get

$$a_0 = 0;\ a_1 = k;\ a_2 = 1 + k\bar{A};\ a_3 = k_0 + 2k\bar{B}$$

and

$$\Delta = kr + (1 + k\bar{A})r^2 + (k_0 + 2k\bar{B})r^3$$

is the reflecting integral of (3.1). Similar to the proof of Theorem 3.2, the first integral of (3.1) is

$$u=\int\frac{1}{kr+(1+k\bar{A})r^2+(k_0+2k\bar{B})r^3}dr,$$

calculating this indefinite integral we get the conclusions of the present theorem. □

Corollary 3.2. *If the conditions of Theorem 2.1 are satisfied and $p_{20}^2+p_{11}^2\neq 0$, then system*

$$\begin{cases}\frac{dx}{dt}=-y+x(P_2(x,y)+P_4(x,y))+\cos\theta\alpha(\theta,u)\Delta,\\ \frac{dy}{dt}=x+y(P_2(x,y)+P_4(x,y))+\sin\theta\alpha(\theta,u)\Delta\end{cases}$$

has a center at (0,0), where

$$P_2(x,y)=p_{20}x^2+p_{11}xy+p_{02}y^2,\ P_4(x,y)=p_{40}x^4+p_{31}x^3y+p_{22}x^2y^2+p_{13}xy^3+p_{04}y^4,$$

$\alpha(\theta,u)$ is an arbitrary continuously differentiable 2π-periodic odd function with respect to θ,

$$\theta=\arctan\frac{y}{x},\ \rho=\sqrt{x^2+y^2},\ \Delta=\rho(k+(1+2k\bar{P}_2)\rho^2+(k_0+4k\bar{P}_4)\rho^4),\ u=\int\Delta^{-1}d\rho,$$

$$P_2=P_2(\cos\theta,\sin\theta),\ P_4=P_4(\cos\theta,\sin\theta),\ \bar{P}_2=\int_0^\theta P_2d\theta,\ \bar{P}_4=\int_0^\theta P_4d\theta,$$

k_0, k are nonzero constants.

Proof. By the assumption and Theorem 2.1, system (2.1) has a center at (0,0) and $P_4=(k_0+2k_1\bar{P}_2)P_2$, *i.e.*, $B=(k_0+k_1\bar{A})A$, $(B=2P_4,\ A=2P_2)$, k_0, k_1 are nonzero constants. Using Theorem 3.3, we get the equation

$$\frac{d\rho}{d\theta}=\rho^3(P_2+P_4\rho^2)$$

is equivalent to equation

$$\frac{d\rho}{d\theta}=\rho^3(P_2+P_4\rho^2)+\alpha(\theta,u)\Delta, \tag{3.11}$$

where

$$\Delta=\rho(k+(1+2k\bar{P}_2)\rho^2+(k_0+4k\bar{P}_4)\rho^4),\ (k=\frac{k_1}{k_0}),\ u=\int\Delta^{-1}d\rho.$$

Similar to the proof of the Corollary 3.1, the present conclusion is true. □

Example 3.2. If the conditions of Theorem 2.1 are satisfied and

$$p_{13}+p_{31}=0,\ p_{11}=0,\ p_{40}+p_{04}=0,\ p_{20}=3, p_{31}=4,$$

by the proof of Theorem 2.1, we have $P_4=P_2(k_0+2k_1\bar{P}_2)$, here, $k_0=\frac{p_{40}}{3}$, $k_1=\frac{2}{9}$. By Theorem 3.3, the equation (2.6) has a reflecting integral

$$\Delta=\rho(2+(p_{40}+\frac{2}{3}\bar{P}_2)\rho^2)(1+(p_{40}+\frac{2}{3}\bar{P}_2)\rho^2)$$

and has a first integral

$$u = \frac{2 + (p_{40} + \frac{2}{3}\bar{P}_2)\rho^2}{(1 + (p_{40} + \frac{2}{3}\bar{P}_2)\rho^2)^2}\rho^2.$$

Taking $\alpha = 2\lambda u \sin\theta\cos\theta\Delta$, ($\lambda$ is a constant), (3.11) becomes

$$\frac{d\rho}{d\theta} = \frac{(P_2\rho^3 + P_4\rho^5)(1 + (p_{40} + \frac{2}{3}\bar{P}_2)\rho^2) + 2\lambda\sin\theta\cos\theta\rho^3(2 + (p_{40} + \frac{2}{3}\bar{P}_2)\rho^2)^2}{1 + (p_{40} + \frac{2}{3}\bar{P}_2)\rho^2},$$

it follows its equivalent system

$$\begin{cases} \frac{dx}{dt} = P(1 + p_{40}(x^2 + y^2) + 2xy + \frac{1}{3}p_{11}y^2) + yx^2\Phi, \\ \frac{dy}{dt} = Q(1 + p_{40}(x^2 + y^2) + 2xy + \frac{1}{3}p_{11}y^2) + y^2x\Phi \end{cases} \tag{3.12}$$

has a center at (0,0), where

$$P = -y + x(P_2(x, y) + P_4(x, y)),\ Q = x + y(P_2(x, y) + P_4(x, y)),$$

$$\Phi = 2\lambda(x^2 + y^2)(2 + p_{40}(x^2 + y^2) + 2xy + \frac{1}{3}p_{11}y^2)^2.$$

System (3.12) has a first integral:

$$H = ue^{\lambda\cos^2\theta} = \frac{2 + p_{40}(x^2 + y^2) + 2xy + \frac{1}{3}p_{11}y^2}{(1 + p_{40}(x^2 + y^2) + 2xy + \frac{1}{3}p_{11}y^2)^2}e^{\frac{\lambda x^2}{x^2+y^2}}(x^2 + y^2).$$

Theorem 3.4. *If $A_j(\theta) = k_jA_2(\theta)\bar{A}_2^{j-2}(\theta)$, $(j = 3, 4, ..., n)$, k_j are constants, then equation*

$$\frac{dr}{d\theta} = A_2(\theta)r^2 + A_3(\theta)r^3 + ... + A_n(\theta)r^n,\ (n \geq 3) \tag{3.13}$$

has a reflecting integral

$$\Delta = r(1 + \bar{A}_2(\theta)r + 2\bar{A}_3(\theta)r^2 + 3\bar{A}_4(\theta)r^3 + ... + (n-1)\bar{A}_n(\theta)r^{n-1}) \tag{3.14}$$

and has a first integral

$$u = \int \frac{1}{r(1 + \bar{A}_2(\theta)r + 2\bar{A}_3(\theta)r^2 + 3\bar{A}_4(\theta)r^3 + ... + (n-1)\bar{A}_n(\theta)r^{n-1})}dr. \tag{3.15}$$

Furthermore, if $A_2(\theta + 2\pi) = A_2(\theta)$ and $\int_0^{2\pi} A_2(\theta)d\theta = 0$, then equation (3.13) and its equivalent equation

$$\frac{dr}{d\theta} = \sum_{i=2}^{n} A_i(\theta)r^i + \alpha(\theta, u)\Delta(\theta, r), \tag{3.16}$$

have a center at $r = 0$, where $\alpha(\theta, u)$ is an arbitrary differentiable odd and 2π-periodic function with respect to θ.

Proof. Since $A_j(\theta) = k_jA_2(\theta)\bar{A}_2^{j-2}(\theta)$, $(j = 3, 4, ..., n)$, so

$$(i-1)(i-j)\bar{A}_iA_j + (j-1)(j-i)\bar{A}_jA_i = 0,\ (i, j = 2, 3, 4, ..., n).$$

By this, it is not difficult to check that function (3.14) is a solution of equation (3.6). Thus, function (3.14) is the reflecting integral of (3.13). Similar to the proof of Theorem 3.2, the first integral of equation (3.13) is (3.15). As $\bar{A}_j$ are 2π-periodic functions , $\Delta(\theta, r)$ and $u(\theta, r)$ are 2π-periodic, thus $r(\theta)$ is 2π-periodic, equation (3.13) and its equivalent equation (3.16) have a center at $r = 0$. □

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