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Finite-time Stability of Nonlinear Fractional Order Systems with a Constant Delay*

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Abstract In this paper, based on the generalized Gronwall inequality and the method of steps, an approach to the finite-time stability of nonlinear fractional order systems with a constant delay is proposed. A sufficient condition for finite-time stability of considered systems is presented. Compared with the finite-time stability criteria in the existing literature, our results are less conservative. Two examples are given to illustrate the effectiveness of the proposed theorem.

Keywords Finite-time stability, Time delay, Generalized Gronwall inequality.

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1. Introduction

Fractional calculus is playing an increasingly key role in science and engineering fields [19]. Some eminent results have been obtained in fractional differential systems recently [7, 12, 25–27]. One of most important tasks in fractional differential systems is stability analysis. Many researchers focus on the asymptotic stable. However, in practical engineering applications, many systems state trajectories must not exceed a certain bound over a given finite-time interval. The systems of short-time working, for example, missile systems, satellite systems, flight control systems [5], robotic manipulation systems [13], are main examples of such applications.

Finite-time stability of fractional delay differential systems is initially investigated by Lazarevic [8, 9], after that time, many authors [6, 16, 21, 23] adopted the similar approach in [9] to study the fractional delay differential systems. However, the proof of the main theorems in these papers contains a flaw, that is, $f(t) = \sup_{-\tau\le\theta\le 0}\|x(t+\theta)\|$ is not always monotone increasing with respect to t, which leads to that $F_1(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds$ is not always monotone increasing with respect to t. As a result, the Gronwall inequality can't be used in this

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case. Recently, some different techniques are developed without the help of $f(t) = \sup_{-\tau\le\theta\le 0}\|x(t+\theta)\|$. In 2018, the authors [22] tried to overcome this difficulty by transforming the delayed term $\|x(t-\tau)\|$ to non-delayed term $\|x(t)\|$. Unfortunately, the obtained result is also incorrect, since the proof [22] is based on that $F_2(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|u(s)\|ds$ is monotone increasing with respect to t, which is not always correct (see [4]). Similar problem exists in [28, Lemma 2.4].

The main difficulty in using Gronwall inequality to investigate the finite-time stability of nonlinear fractional order systems with time delays is in estimating the delayed term. In [17], Phat et al. overcame this difficulty successfully by constructing a new auxiliary function $u(t) = \sup_{\theta\in[-h,t]}\|x(\theta)\|$. Some different approaches, without using Gronwall inequality to investigate the stability of fractional with time delays, were presented in [10, 11] based on delayed Mittag-Leffler type matrix. In addition, Thanh et al. [20] proposed an approach based on the Laplace transform and a independent-delay Lyapunov functional $V(x(t))$ to study finite-time stability of nonlinear fractional-order systems with interval time-varying delay. In [1–3], the Holder inequality was introduced to guarantee that fractional order delayed system is finite-time stable. The above discussions inspire us for the present investigation.

In this paper, we adopt the method of steps and the generalized Gronwall inequality to give a sufficient condition for the finite-time stability of nonlinear fractional order delay system

$$\begin{cases} {}_0^C D_t^\alpha x(t) = A_0 x(t) + A_1 x(t-\tau) + f(t, x(t), x(t-\tau)), t\in[0,T], \\ x(t) = \varphi(t), t\in[-\tau, 0]. \end{cases} \tag{1.1}$$

The rest of the paper is organized as follows. In Section 2, some definitions and properties of the fractional delay differential equations are introduced. In Section 3, a proof of uniqueness theorem of nonlinear fractional-order time varying delay system as an extension to [21, Theorem 3.2] is given. In Section 4, a gap existing in the proof of [21, Theorem 3.3] is pointed out. Moreover, a new estimate value of the solution of the systems (1.1) is given and a sufficient condition for finite-time stability of the systems (1.1) is presented. In Section 5, two examples are given to illustrate our results.

2. Preliminaries

In this section, we give some basic definitions and notations. Let $\|x(t)\|_1$ be the 1-norm of a vector $x(t)\in R^n$, where $\|x(t)\|_1 = \sum_{i=1}^n |x_i(t)|$. The induced norm $\|A\|_1$ of the matrix A is defined as $\|A\|_1 = \max_{1\le j\le n}\sum_{i=1}^n |a_{i,j}|$. Throughout this paper, all the notation $\|\cdot\|$ means $\|\cdot\|_1$.

Definition 2.1. [18] For $x(t)\in L^1([0,+\infty), R)$, the Riemann-Liouville integral of order α for x is defined by

$${}_0I_t^\alpha x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds,$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. [18] For $x(t) \in C^1([0,+\infty), R)$, the Caputo derivative of order α for x is defined by

$$ {}_0^C D_t^\alpha x(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x'(s)ds, \quad \alpha \in (0,1]. $$

Consider nonlinear fractional order delay system

$$ \begin{cases} {}_0^C D_t^\alpha x(t) = A_0 x(t) + A_1 x(t-\tau) + f(t, x(t), x(t-\tau)), t \in [0,T], \\ x(t) = \varphi(t), t \in [-\tau, 0], \end{cases} \tag{2.1} $$

where ${}_0^C D_t^\alpha$ denotes the Caputo fractional derivative of order $\alpha \in (0,1)$, $x(t) \in R^n$ is the state vector of the system, $A_0, A_1 \in R^{n\times n}$ are constant matrices, τ is a constant delay, $f(t, x(t), x(t-\tau)) \in C([0,T] \times R^n \times R^n, R^n)$ is a nonlinear function, $\varphi \in C([-\tau, 0], R^n)$ is the initial function with the norm $\|\varphi\|_c = \sup_{t\in[-\tau,0]} \|\varphi(t)\|$. We make the following assumptions.

(H1) For $f(t, x(t), x(t-\tau)) \in C([0,T] \times R^n \times R^n, R^n)$, there exists a nonnegative nondecreasing continuous function $L(t)$ such that

$$ \begin{aligned} \|f(t, x_1(t), x_1(t-\tau)) - f(t, x_2(t), x_2(t-\tau))\| \le & L(t)(\|x_1(t) - x_2(t)\| \\ & + \|x_1(t-\tau) - x_2(t-\tau)\|) \end{aligned} $$

for any $t \in [0,T]$.

(H2) $f(t, 0, 0) = [\underbrace{0, 0, \cdots, 0}_{n}]^T$.

Definition 2.3. The system (2.1) is finite-time stable w.r.t.$\{\delta, \varepsilon, T\}$ with $\delta < \varepsilon$ if and only if $\|\varphi\|_c \le \delta$ implies $\|x(t)\| \le \varepsilon, \forall t \in [0,T]$.

Lemma 2.1 ([24]). (*Generalized Gronwall Inequality*) *Suppose* $\alpha > 0$, $f(t)$ *is a nonnegative, nondecreasing function locally integrable on* $[0,T)$ (*some* $T \le +\infty$) *and* $g(t)$ *is a nonnegative, nondecreasing continuous function defined on* $[0,T)$, $g(t) \le M$(*constant*), *and suppose* $x(t)$ *is nonnegative and locally integrable on* $[0,T)$ *with*

$$ x(t) \le f(t) + g(t)_0 I_t^\alpha x(t) $$

on this interval. Then

$$ x(t) \le f(t) E_\alpha(g(t)t^\alpha), \quad t \in [0,T), $$

where E_α *is the Mittag-Leffler function defined by* $E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\alpha+1)}$.

Lemma 2.2. *Assume* $0 < g \le f$, $\alpha \in (0,1]$. *Then*

$$ f^\alpha - g^\alpha \le (f-g)^\alpha. \tag{2.2} $$

Proof. The proof is easy, so it is omitted here. □

3. Uniqueness Theorem of Nonlinear Fractional-Order Time Varying Delay System

Consider the system

$$ \begin{cases} {}_0^C D_t^\alpha x(t) = A_0 x(t) + A_1 x(t-\tau(t)) + f(t, x(t), x(t-\tau(t))), t \in [0,T], \\ x(t) = \varphi(t), t \in [-\tau, 0], \end{cases} \tag{3.1} $$

where $\tau(t)$ is a continuous function satisfying $0 \leq \tau(t) \leq \tau, t \geq 0$, other notations have the same meaning with ones in the system (2.1). We make the following assumption.

(H3) For $f(t, x(t), x(t-\tau(t))) \in C([0,T] \times R^n \times R^n, R^n)$, there exists a positive constant $L > 0$ such that

$$\begin{aligned}&\|f(t, x_1(t), x_1(t-\tau(t))) - f(t, x_2(t), x_2(t-\tau(t)))\| \\ \leq & L(\|x_1(t) - x_2(t)\| + \|x_1(t-\tau(t)) - x_2(t-\tau(t))\|)\end{aligned}$$

for any $t \in [0,T]$.

The research of the existence and uniqueness of the solution to the system is the basis of its stability investigation. Motivated by [17], we give the proof of uniqueness theorem for the system (3.1) using the generalized Gronwall inequality .

Theorem 3.1. *Assume that* $(H3)$ *holds. Then the system* (3.1) *has a unique continuous solution.*

Proof. Let $x(t)$ and $\tilde{x}(t)$ be any two different solutions to the system (3.1). We define $z(t) = \tilde{x}(t) - x(t)$. If $t \in [-\tau, 0]$, then $z(t) = 0$, for $t \in [-\tau, 0]$. That is to say, the system (3.1) has a unique solution for $t \in [-\tau, 0]$.

If $t \in [0,T]$, applying the Riemann-Liouville fractional integral ${}_0I_t^\alpha$ on both sides of the system (3.1), then

$$\begin{aligned} z(t) = & {}_0I_t^\alpha[A_0 z(t) + f(t, x(t), x(t-\tau(t))) - f(t, \tilde{x}(t), \bar{x}(t-\tau(t))))] \\ & + {}_0I_t^\alpha[A_1 z(t-\tau(t))]. \end{aligned} \tag{3.2}$$

Applying the norm on both sides of (3.2), it follows that

$$\begin{aligned} \|z(t)\| \leq & {}_0I_t^\alpha[\|A_0\|\|z(t)\| + \|f(t, x(t), x(t-\tau(t)) - f(t, \tilde{x}(t), \bar{x}(t-\tau(t))\|] \\ & + {}_0I_t^\alpha[\|A_1\|\|z(t-\tau(t))\|] \\ \leq & (\|A_0\| + L){}_0I_t^\alpha\|z(t)\| + (\|A_1\| + L){}_0I_t^\alpha\|z(t-\tau(t))\|. \end{aligned} \tag{3.3}$$

Let $z^*(t) = \sup_{\theta \in [-\tau, t]} \|z(\theta)\|$ for $t \in [0,T]$, it is obvious that $z^*(t)$ is increasing with respect to t and we have

$$\|z(t-\tau(t))\| \leq z^*(t)$$

and

$$\|z(t)\| \leq z^*(t).$$

It follows from (3.3) that

$$\begin{aligned} \|z(t)\| &\leq (\|A_0\| + L){}_0I_t^\alpha z^*(t) + (\|A_1\| + L){}_0I_t^\alpha z^*(t) \\ &= (\|A_0\| + \|A_1\| + 2L){}_0I_t^\alpha z^*(t) \end{aligned}$$

Note that for all $\theta \in [0,t]$, we have

$$\|z(\theta)\| \leq (\|A_0\| + \|A_1\| + 2L){}_0I_\theta^\alpha z^*(\theta).$$

Let $Z(t) = {}_0I_t^\alpha z^*(t)$, then

$$Z(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} z^*(s) ds$$

$$= \int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} z^*(t-s)ds.$$

Since

$$\begin{aligned} Z'(t) &= \int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} \frac{dz^*(t-s)}{dt} ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} z^*(0) \\ &= \int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} \frac{dz^*(t-s)}{dt} ds \\ &\geq 0. \end{aligned}$$

We have the function $Z(t) = {}_0I_t^\alpha z^*(t)$ is increasing respect to t, and hence

$$\|z(\theta)\| \leq (\|A_0\| + \|A_1\| + 2L)_0 I_t^\alpha z^*(t), \qquad \theta \in [0, t].$$

Therefore, we have

$$\begin{aligned} z^*(t) &= \sup_{\theta \in [-\tau, t]} \|z(\theta)\| \\ &\leq \max\left\{ \sup_{\theta \in [-\tau, 0]} \|z(\theta)\|, \sup_{\theta \in [0, t]} \|z(\theta)\| \right\} \\ &= \max\{0, (\|A_0\| + \|A_1\| + 2L)_0 I_t^\alpha z^*(t)\} \\ &= (\|A_0\| + \|A_1\| + 2L)_0 I_t^\alpha z^*(t) \end{aligned} \tag{3.4}$$

Applying Lemma 2.1 on (3.4), it follows that

$$\|z(t)\| \leq z^*(t) \leq 0 \cdot E_\alpha[(\|A_0\| + \|A_1\| + 2L)t^\alpha].$$

Hence we obtain $x(t) = \tilde{x}(t)$ for $t \in [0, T]$. This completes the proof. □

Remark 3.1. If $\tau(t) = \tau$ is a constant, it follows from Theorem 3.1 that the solution of system (2.1) is unique.

Remark 3.2. In [17], new criteria for finite-time stability of nonlinear fractional order **time varying delay** system was given. This criteria is based on corresponding existence and uniqueness theorem ([21, Theorem 3.2]). However, there is a flaw in [21, Theorem 3.2]. A rigorous proof of the uniqueness theorem can be seen in [4].

Remark 3.3. We have to point out that [21, Theorem 3.2] was proved for system (2.1) which is nonlinear fractional order system with a **constant delay**. Since $t \in [0, \tau]$ can't imply $t - \tau(t) \in [-\tau, 0]$, in other words, $x(t - \tau(t)) \neq 0$ for $t \in [0, \tau]$, the method in [4] is not valid for the proof of the uniqueness theorem of the system (3.1). Therefore, our uniqueness result in this paper is more general than one in [4].

Remark 3.4. The assumption (2) in [17] can't imply the assumption (H3) in this paper, in other words, uniqueness theorem of the system (3.1) can't hold under the assumption (2) in [17].

4. Finite-Time Stability of the System (2.1)

In [21, Theorem 3.3], the authors obtained the estimate value of the solution by using the generalized Gronwall inequality. In this section, we will show that the result is not always correct and give a new estimate value of the solution.

By setting $u(t) = \|\varphi\|_c + \frac{M+\|\varphi\|_c(\|A_0\|+\|A_1\|)}{\Gamma(\alpha+1)}t^\alpha$, the authors [21] applied the Lemma 2.1 for the following inequality:

$$\|x(t)\| \le \|\varphi\|_c + \frac{M + \|\varphi\|_c(\|A_0\| + \|A_1\|)}{\Gamma(\alpha+1)}t^\alpha + (\|A_0\| + \|A_1\|)_0 I_t^\alpha \sup_{\theta\in[-\tau,0]} \|x(t+\theta)\|$$

and had the statement

$$\|x\| \le \sup_{\theta\in[-\tau,0]} \|x(t+\theta)\| \le u(t)E_\alpha[(\|A_0\| + \|A_1\|)t^\alpha], \text{ see [21, (22)].} \tag{4.1}$$

However, we have to point out the statement (4.1) is not always correct since ${}_0I_t^\alpha x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds$ is not always monotone increasing with respect to t for any $x(t) \ge 0$. A counter example was given in [4]. Furthermore, the authors [4] have proved that ${}_0I_t^{0.5}\frac{1}{t+1} = \int_0^t \frac{(t-s)^{-0.5}}{\Gamma(0.5)}\frac{1}{s+1}ds$ is monotone decreasing for large t.

Now a question naturally arises, how to give a correct estimate value of the solution of the system (2.1)? Based on the generalized Gronwall inequality and the method of steps [14], we give the estimate of the solution and further obtain the finite-time stability of the system.

Let $T > \tau$, $[0, T] = \bigcup_{i=0}^n [i\tau, (i+1)\tau] \bigcup [(n+1)\tau, T]$, with $(n+1)\tau < T \le (n+2)\tau$. $\sup_{s\in[0,t]}(\|A_0\| + L(s)) = \beta_0(t)$, $\sup_{s\in[0,t]}(\|A_1\| + L(s)) = \beta_1(t)$.

Theorem 4.1. *Assume that the following conditions* $(H1)$, $(H2)$ *hold. Then the solution of the system* (2.1) *is finite-time stable w.r.t* $\{\delta, \varepsilon, T\}$, *if the following conditions is satisfied:*

$$w_T(\tau)E_\alpha(\beta_0(T)T^\alpha) \le \varepsilon, \tag{4.2}$$

where

$$\begin{aligned}
w_T(\tau) =& \left(1 + \beta_1(T)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right)\delta \\
&+ \beta_1(T)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\left[\sum_{j=1}^n w_j(\tau)E_\alpha(\beta_0(j\tau)(j\tau)^\alpha)\right] \\
&+ \beta_1(T)\frac{(T-(n+1)\tau)^\alpha}{\Gamma(\alpha+1)}w_{n+1}(\tau)E_\alpha(\beta_0((n+1)\tau)((n+1)\tau)^\alpha), \\
w_{i+1}(\tau) =& \left(1 + \beta_1((i+1)\tau)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right)\delta \\
&+ \beta_1((i+1)\tau)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\left[\sum_{j=1}^i w_j(\tau)E_\alpha(\beta_0(j\tau)(j\tau)^\alpha)\right],\ 1 \le i \le n, \\
w_1(\tau) =& \delta + \beta_1(\tau)\delta\frac{\tau^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

Proof. From [21, Theorem 3.1], we know for $t \in [0, T]$, we have

$$x(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[A_0x(s) + A_1x(s-\tau) + f(s, x(s), x(s-\tau))]ds. \tag{4.3}$$

Applying the norm on both sides of (4.3), for $t \in [0, \tau]$, we have

$$\|x(t)\| \le \|\varphi(0)\| + {}_0I_t^\alpha\|A_0x(t) + A_1x(t-\tau) + f(t, x(t), x(t-\tau))\|$$

$$
\begin{aligned}
&\leq \|\varphi\|_c + \|A_0\| {}_0I_t^\alpha \|x(t)\| + \|A_1\| {}_0I_t^\alpha \|x(t-\tau)\| \\
&\quad + {}_0I_t^\alpha \|f(t, x(t), x(t-\tau))\| \\
&\leq \|\varphi\|_c + \|A_0\| {}_0I_t^\alpha \|x(t)\| + \|A_1\| {}_0I_t^\alpha \|x(t-\tau)\| \\
&\quad + {}_0I_t^\alpha [L(t)(\|x(t)\| + \|x(t-\tau)\|)] \\
&= \|\varphi\|_c + \beta_0(t) {}_0I_t^\alpha \|x(t)\| + \beta_1(t) {}_0I_t^\alpha \|x(t-\tau)\| \\
&= \|\varphi\|_c + \beta_0(t) {}_0I_t^\alpha \|x(t)\| + \beta_1(t) \|\varphi\|_c \frac{t^\alpha}{\Gamma(\alpha+1)} \\
&\leq \left[\|\varphi\|_c + \beta_1(t) \|\varphi\|_c \frac{t^\alpha}{\Gamma(\alpha+1)} \right] + \beta_0(t) {}_0I_t^\alpha \|x(t)\|.
\end{aligned}
$$

Using the Lemma 2.1 for $t \in [0, \tau]$, we have

$$
\begin{aligned}
\|x(t)\| &\leq \left[\|\varphi\|_c + \beta_1(t) \|\varphi\|_c \frac{t^\alpha}{\Gamma(\alpha+1)} \right] E_\alpha(\beta_0(t) t^\alpha) \\
&\leq \left[\|\varphi\|_c + \beta_1(\tau) \|\varphi\|_c \frac{\tau^\alpha}{\Gamma(\alpha+1)} \right] E_\alpha(\beta_0(\tau) \tau^\alpha) \\
&\leq w_1(\tau) E_\alpha(\beta_0(\tau) \tau^\alpha).
\end{aligned}
$$

For $t \in (i\tau, (i+1)\tau]$, $1 \leq i \leq n$, we have

$$
\begin{aligned}
\|x(t)\| &\leq \|\varphi\|_c + \beta_0(t) {}_0I_t^\alpha \|x(t)\| + \frac{\beta_1(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x(s-\tau)\| ds \\
&= \|\varphi\|_c + \beta_0(t) {}_0I_t^\alpha \|x(t)\| + \beta_1(t) \left[\int_0^\tau \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|x(s-\tau)\| ds \right. \\
&\quad + \int_\tau^{2\tau} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|x(s-\tau)\| ds \\
&\quad + \cdots \\
&\quad \left. + \int_{i\tau}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|x(s-\tau)\| ds \right] \\
&\leq \|\varphi\|_c + \beta_0(t) {}_0I_t^\alpha \|x(t)\| + \beta_1(t) \left[\|\varphi\|_c \frac{t^\alpha - (t-\tau)^\alpha}{\Gamma(\alpha+1)} \right. \\
&\quad + w_1(\tau) E_\alpha(\beta_0(\tau)\tau^\alpha) \frac{(t-\tau)^\alpha - (t-2\tau)^\alpha}{\Gamma(\alpha+1)} \\
&\quad + \cdots \\
&\quad \left. + w_i(\tau) E_\alpha(\beta_0(i\tau)(i\tau)^\alpha) \frac{(t-i\tau)^\alpha}{\Gamma(\alpha+1)} \right] \\
&\overset{(2.2)}{\leq} \|\varphi\|_c + \beta_0(t) {}_0I_t^\alpha \|x(t)\| + \beta_1(t) \left[\|\varphi\|_c \frac{\tau^\alpha}{\Gamma(\alpha+1)} \right. \\
&\quad + w_1(\tau) E_\alpha(\beta_0(\tau)\tau^\alpha) \frac{\tau^\alpha}{\Gamma(\alpha+1)} \\
&\quad + \cdots \\
&\quad \left. + w_i(\tau) E_\alpha(\beta_0(i\tau)(i\tau)^\alpha) \frac{\tau^\alpha}{\Gamma(\alpha+1)} \right] \\
&\leq \left(1 + \beta_1((i+1)\tau) \frac{\tau^\alpha}{\Gamma(\alpha+1)} \right) \|\varphi\|_c
\end{aligned}
$$

$$
\begin{aligned}
&+\beta_1((i+1)\tau)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\left[\sum_{j=1}^{i} w_j(\tau)E_\alpha(\beta_0(j\tau)(j\tau)^\alpha)\right]\\
&+\beta_0((i+1)\tau)_0I_t^\alpha\|x(t)\|.
\end{aligned}
$$

Using Lemma 2.1 for $t\in(i\tau,(i+1)\tau]$, we have

$$
\begin{aligned}
\|x(t)\| &\le w_{i+1}(\tau)E_\alpha(\beta_0((i+1)\tau)t^\alpha)\\
&\le w_{i+1}(\tau)E_\alpha(\beta_0((i+1)\tau)((i+1)\tau))^\alpha)
\end{aligned}
$$

Finally, for $t\in((n+1)\tau,T]$, we have

$$
\begin{aligned}
\|x(t)\| \le&\|\varphi\|_c+\beta_0(t)_0I_t^\alpha\|x(t)\|+\frac{\beta_1(t)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|x(s-\tau)\|ds\\
=&\|\varphi\|_c+\beta_0(t)_0I_t^\alpha\|x(t)\|+\beta_1(t)\left[\int_0^\tau \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|x(s-\tau)\|ds\right.\\
&+\int_\tau^{2\tau}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|x(s-\tau)\|ds\\
&+\cdots\\
&+\int_{n\tau}^{(n+1)\tau}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|x(s-\tau)\|ds\\
&\left.+\int_{(n+1)\tau}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|x(s-\tau)\|ds\right]\\
\le&\|\varphi\|_c+\beta_0(t)_0I_t^\alpha\|x(t)\|+\beta_1(t)\left[\|\varphi\|_c\frac{t^\alpha-(t-\tau)^\alpha}{\Gamma(\alpha+1)}\right.\\
&+w_1(\tau)E_\alpha(\beta_0(\tau)\tau^\alpha)\frac{(t-\tau)^\alpha-(t-2\tau)^\alpha}{\Gamma(\alpha+1)}\\
&+\cdots\\
&+w_n(\tau)E_\alpha(\beta_0(n\tau)(n\tau)^\alpha)\frac{(t-n\tau)^\alpha-(t-(n+1)\tau)^\alpha}{\Gamma(\alpha+1)}\\
&\left.+w_{n+1}(\tau)E_\alpha(\beta_0((n+1)\tau)((n+1)\tau)^\alpha)\frac{(t-(n+1)\tau)^\alpha}{\Gamma(\alpha+1)}\right]\\
\le&\|\varphi\|_c+\beta_0(t)_0I_t^\alpha\|x(t)\|+\beta_1(t)\left[\|\varphi\|_c\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right.\\
&+w_1(\tau)E_\alpha(\beta_0(\tau)\tau^\alpha)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\\
&+\cdots\\
&+w_n(\tau)E_\alpha(\beta_0(n\tau)(n\tau)^\alpha)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\\
&\left.+w_{n+1}(\tau)E_\alpha(\beta_0((n+1)\tau)((n+1)\tau)^\alpha)\frac{(t-(n+1)\tau)^\alpha}{\Gamma(\alpha+1)}\right]\\
\le&\left(1+\beta_1(T)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right)\|\varphi\|_c\\
&+\beta_1(T)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\left[\sum_{j=1}^{n} w_j(\tau)E_\alpha(\beta_0(j\tau)(j\tau)^\alpha)\right]
\end{aligned}
$$

$$+ \beta_1(T)\frac{(T-(n+1)\tau)^\alpha}{\Gamma(\alpha+1)}w_{n+1}(\tau)E_\alpha(\beta_0((n+1)\tau)((n+1)\tau)^\alpha)$$
$$+ \beta_0(T)_0I_t^\alpha\|x(t)\|.$$

Using Lemma 2.1 for $t \in ((n+1)\tau, T]$, we have

$$\|x(t)\| \le w_T(\tau)E_\alpha(\beta_0(T)T^\alpha).$$

□

Remark 4.1. If $T \in (0,\tau]$, then the system (2.1) is finite-time stable under the following condition:

$$w_T(\tau)E_\alpha(\beta_0(T)T^\alpha) \le \varepsilon,$$

where

$$w_T(\tau) = \delta + \beta_1(T)\frac{\delta T^\alpha}{\Gamma(\alpha+1)}.$$

Remark 4.2. A similar approach is used to study the finite-time stability of linear fractional order time delay system in [15]. However, we have to point out flaws in [15]. For $t \in [0, 2\tau]$, The authors [15] estimated the term

$$\frac{v_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|x(s-\tau)\|ds \quad [15, (9)] \tag{4.4}$$

using the estimate value of $x(t)$ for $t \in [0,\tau]$, but $s-\tau \in [-\tau,\tau]$, so the authors can't estimate the term (4.4) by use of the estimate value of $x(t)$ for $t \in [0,\tau]$ directly, namely, we have to split this term into two terms:

$$\frac{v_1}{\Gamma(\alpha)}\int_0^\tau (t-s)^{\alpha-1}\|x(s-\tau)\|ds \tag{4.5}$$

and

$$\frac{v_1}{\Gamma(\alpha)}\int_\tau^t (t-s)^{\alpha-1}\|x(s-\tau)\|ds. \tag{4.6}$$

In this way, for $s \in [\tau, t]$, (4.6) can be estimated by the estimate value of $x(t)$ for $t \in [0,\tau]$. For $s \in [0,\tau]$, (4.5) can be estimated by $\|x(s-\tau)\| \le \|\varphi\|_c$. Furthermore, (4.4) can be estimated. Similar problems exist in the estimate of $x(t)$ for $t \in [0,T]$ [15].

Corollary 4.2. The solution of the following linear fractional order system with a constant delay

$$\begin{cases} {}_0^C D_t^\alpha x(t) = A_0x(t) + A_1x(t-\tau), t \in [0,T], \\ x(t) = \varphi(t), t \in [-\tau, 0] \end{cases} \tag{4.7}$$

is finite-time stable w.r.t $\{\delta, \varepsilon, T\}$, if the following conditions is satisfied:

$$w_T(\tau)E_\alpha(\|A_0\|T^\alpha) \le \varepsilon, \tag{4.8}$$

where

$$w_T(\tau) = \left(1 + \|A_1\|\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right)\delta$$

$$
\begin{aligned}
&+\|A_1\|\frac{\tau^\alpha}{\Gamma(\alpha+1)}\left[\sum_{j=1}^{n} w_j(\tau)E_\alpha(\|A_0\|(j\tau)^\alpha)\right]\\
&+\|A_1\|\frac{(T-(n+1)\tau)^\alpha}{\Gamma(\alpha+1)}w_{n+1}(\tau)E_\alpha(\|A_0\|((n+1)\tau)^\alpha),\\
w_{i+1}(\tau) =&\left(1+\|A_1\|\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right)\delta\\
&+\|A_1\|\frac{\tau^\alpha}{\Gamma(\alpha+1)}\left[\sum_{j=1}^{i} w_j(\tau)E_\alpha(\|A_0\|(j\tau)^\alpha)\right],\ 1\le i\le n,\\
w_1(\tau) =&\delta+\|A_1\|\delta\frac{\tau^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}
$$

Remark 4.3. If $T \in (0, \tau]$, then the system (4.7) is finite-time stable under the following condition:

$$w_T(\tau)E_\alpha(\|A_0\|T^\alpha) \le \varepsilon,$$

where

$$w_T = \delta + \delta\frac{\|A_1\|T^\alpha}{\Gamma(\alpha+1)}.$$

Remark 4.4. The criterion for finite-time stability of system (4.7) from [17, Theorem 1] is given as follows:

$$\delta E_\alpha[(\|A_0\| + \|A_1\|)T^\alpha] \le \varepsilon.$$

Remark 4.5. The criterion for finite-time stability of system (2.1) from [17, Theorem 1] is given as follows:

$$\delta E_\alpha[(\|A_0\| + \|A_1\| + 2L)T^\alpha] \le \varepsilon.$$

5. Numerical examples

In this section, we will use two modified examples which appeared originally in [9] and [17], respectively, to show the effectiveness of Theorem 4.1 and Corollary 4.2 and to illustrate our results are less conservative than the existing results.

Example 5.1. Consider the linear fractional delay system

$$
{}_0^C D_t^{1/2} x(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1(t-0.1) \\ x_2(t-0.1) \end{bmatrix} \tag{5.1}
$$

with initial condition

$$x(t) = \varphi(t) = (0.01, 0.01)^T, \quad t \in [-0.1, 0].$$

We have $\|A_0\| = 2$, $\|A_1\| = 4$, $\|\varphi\|_c = 0.02 < \delta = 0.03$, $\tau = 0.1$. According to Corollary 4.2, we have the following Table 1.

Table 1. ε for $\delta = 0.03$ and T varies in Example 5.1.

T	0.1	0.15	0.2	0.25	0.3
Remark 4.4	2.1879	13.2776	80.3599	486.1797	2.9412e+03
Corollary 4.2	0.1770	0.7910	1.2992	8.1978	13.5959

Table 1 illustrates our results (Corollary 4.2) are less conservative than ones in Remark 4.4 when T varies within a certain interval.

Example 5.2. Consider the nonlinear fractional delay system

$$ {}_0^C D_t^{1/2} x(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1(t-0.1) \\ x_2(t-0.1) \end{bmatrix} + f(t, x(t), x(t-\tau)) \quad (5.2) $$

with initial condition

$$ \varphi(t) = (0.07, 0.07)^T, \ t \in [-0.1, 0], $$

where $f(t, x(t), x(t-\tau)) = 0.01 \begin{bmatrix} \sin x_1(t) \\ \sin x_2(t) \end{bmatrix}$.

We have

$$ \begin{aligned} & \|f(t, x(t), x(t-\tau)) - f(t, y(t), y(t-\tau))\| \\ =& 0.01|\sin x_1(t) - \sin y_1(t)| + 0.01|\sin x_2(t) - \sin y_2(t)| \\ \leq& 0.01|x_1(t) - y_1(t)| + 0.01|x_2(t) - y_2(t)| \\ =& 0.01(\|x(t) - y(t)\| + \|x(t-0.1) - y(t-0.1)\|). \end{aligned} $$

It is easy to see $L(t) = 0.01$, $\|A_0\| = 2$, $\|A_1\| = 4$, $\tau = 0.1$, $\|\varphi\|_c = 0.14 < \delta = 0.15$, $\beta_0(t) = 2.01$, $\beta_1(t) = 4.01$. According to Theorem 4.1 , we have the following Table 2.

Table 2. ε for $\delta = 0.15$ and T varies in Example 5.2.

T	0.3	0.6	0.9	1
Remark 4.5	1.5806e+04	8.3277e+08	4.3876e+13	1.6449e+15
Theorem 4.1	70.0989	8.1581e+05	3.2595e+11	5.3382e+13

Table 2 illustrates our results (Theorem 4.1) are less conservative than ones in Remark 4.5 when T varies within a certain interval.

6. Conclusions

In this paper, the finite-time stability of fractional differential systems with a constant delay have been investigated. By use of the generalized Gronwall inequality and the method of steps, a sufficient condition for finite-time stability of fractional differential system with a constant delay has been proposed. Two examples are given to show that our results are less conservative than the existing ones. It is worth noting that though the authors of [17, 20] investigated the finite-time stability of fractional differential system with interval time-varying delay, two delay functions both are bounded. To the best of our knowledge, finite-time stability of fractional differential system with unbounded delay is still an open problem.

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